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Problem 898. Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain. Let $a, b, c > 0$ and $a + b + c = 3$. Prove that

$$\sum_{\text{cycl}} \frac{bc}{\sqrt{a^2 + 3}} \leq \frac{3\sqrt{2}}{2}$$

where the sum is over all cyclic permutation of $(a, b, c)$ and equality occurs when $a = b = c = 1$.

Solution: First let us observe that $x^2 + 3 \geq \frac{(x + 3)^2}{4}$, which is equivalent to $(x-1)^2 \geq 0$. Using this inequality for the denominators in (1), we observe that (1) follows from the following inequality:

$$\sum_{\text{cycl}} \frac{bc}{\sqrt{a + 3}} \leq \frac{3}{2}$$

One more application of the same idea shows that, (2) follows from the inequality:

$$\sum_{\text{cycl}} \frac{2bc}{\sqrt{a + 3}} \leq \frac{3}{2}$$

So, we will concentrate in proving (3) which is a stronger inequality than the one required in the problem.
Multiplying (3) by 6 and substituting $9 = (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$, (3) changes into
\[
a^2 + b^2 + c^2 - 2ab - 2bc - 2ac + \sum_{\text{cyclic}} 4bc(1 - \frac{3}{\sqrt{a} + 3}) \geq 0 \iff 
\]
\[
a^2 + b^2 + c^2 - 2ab - 2bc - 2ac + 4abc \sum_{\text{cyclic}} \frac{1}{\sqrt{a}(\sqrt{a} + 3)} \geq 0 \iff 
\]
\[
a^2 + b^2 + c^2 - 2ab - 2bc - 2ac + 3abc + 4abc \left( \sum_{\text{cyclic}} \frac{1}{\sqrt{a}(\sqrt{a} + 3)} - \frac{3}{4} \right) \geq 0. \quad (4)
\]
In order to prove (4), we will show the following two inequalities:
\[
a^2 + b^2 + c^2 - 2ab - 2bc - 2ac + 3abc \geq 0, \quad \text{and} \quad (5)
\]
\[
\sum_{\text{cyclic}} \frac{1}{\sqrt{a}(\sqrt{a} + 3)} \geq \frac{3}{4}. \quad (6)
\]
To prove (6), we consider the function $g(x) = \frac{1}{\sqrt{x}(\sqrt{x} + 3)}$ defined for $x > 0$. One can check that the second derivative of $g$ is given by
\[
g''(x) = \frac{1}{4}(8x + 27\sqrt{x} + 27)(\sqrt{x} + 3)^{-3}x^{-5/2} > 0, \quad x > 0.
\]
This shows that $g$ is a convex function and so \( \frac{1}{3}(g(a) + g(b) + g(c)) \geq g(\frac{a+b+c}{3}) = g(1) = \frac{1}{4}. \) This last inequality is equivalent with (6).

Now, to show (5), let us observe that this inequality is equivalent to
\[
(a + b + c)(a^2 + b^2 + c^2 - 2ab - 2bc - 2ac) + 9abc \geq 0 \iff 
\]
\[
a^3 + b^3 + c^3 - a^2(b + c) - b^2(a + c) - c^2(a + b) + 3abc \geq 0. \quad (7)
\]
Without loss of generality let us assume that $a \leq b \leq c$. Since $3 = a + b + c \geq 3a$, we see that $a \leq 1$. Now, (7) can be written as
\[
a(b - a)(c - a) + (b - c)^2(b + c - a) \geq 0 \text{ or } 
\]
\[
a(b - a)(c - a) + (b - c)^2(3 - 2a) \geq 0,
\]
which is true under our assumption. \quad \blacksquare