**Problem 947**: Find all non-constant polynomials $P$ and $Q$ such that

\[ \prod_{i=1}^{n} P(i) = Q \left( \prod_{i=1}^{n} i \right), \]

for all integers $n \geq 1$.

Solution:

The only polynomials with the desired properties are $P(x) = Q(x) = x^m$ for any positive integer $m$.

To see this, let $k$ and $m$ be positive integers and denote $P(x) = p_k x^k + \ldots + p_0$ with $p_k \neq 0$ and $Q(x) = q_m x^m + \ldots + q_0$ with $q_m \neq 0$. Now, since $P(n) = \frac{Q(n!)}{Q((n-1)!)^m}$ for $n \geq 2$,

\[ \lim_{n \to \infty} \frac{Q(n!)}{P(n) Q((n-1)!)} = \lim_{n \to \infty} \frac{q_m(n!)^m}{p_k(n)^k} = \lim_{n \to \infty} \frac{n^m}{p_k(n)^k} = 1. \]

So, $p_k = 1$ and $m = k$. Now, since $P(x) = x^m + (p_{m-1})x^{m-1} + \ldots + p_0$,

\[ \lim_{n \to \infty} [(p_{m-1})x^{m-1} + \ldots + p_0] = \lim_{n \to \infty} [P(n) - n^m] = \lim_{n \to \infty} \left[ \frac{Q(n!)}{Q((n-1)!)^m} - n^m \right] = 0. \]

\[ \lim_{n \to \infty} \left[ \frac{Q(n!)}{Q((n-1)!)^m} \right] = \lim_{n \to \infty} \frac{(q_{m-1})(1-n)(n!)^{m-1} + \ldots + (1-n^m)q_0}{(q_m)(n!)^m + \ldots + q_0} = 0. \]

Thus, $p_0 = \ldots = p_{m-1} = 0$, which yields $P(x) = x^m$. Now, $P(1)P(2) \cdots P(n) = (n!)^m = Q(n!)$, so $Q(x) = x^m$ for infinitely many values of $x$. This is enough to conclude that $Q(x) - x^m$ must be the zero polynomial. Therefore, $P(x) = Q(x) = x^m$. 
