ON THE INNER AUTOMORPHISMS OF FINITE TRANSFORMATION SEMIGROUPS

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If the group of inner automorphisms of a semigroup $S$ of transformations of a finite $n$-element set contains an isomorphic copy of the alternating group $\text{Alt}_n$, then $S$ is an $S_n$-normal semigroup and all the automorphisms of $S$ are inner.


1. Introduction

Given a semigroup $S$ of transformations of a set $X_n = \{1, 2, \ldots, n\}$, denote by $G_S$ the subgroup of the symmetric group $S_n$ of all the permutations $h$ of $X_n$ satisfying $hS = S$. Therefore for each $h \in G_S$, the mapping $\phi_h : S \rightarrow S$ defined by $\phi_h(x) = hxh^{-1}$, for $x \in S$, is an automorphism of $S$. Such an automorphism of $S$ is termed inner [5] and the set of all inner automorphisms of $S$, $\text{Inn} S = \{\phi_h, h \in G_S\}$, forms a subgroup of the group $\text{Aut} S$ of all automorphisms of $S$.

Observe that if $S = T_n$, the semigroup of all total transformations of $X$, then $G_S = S_n$. A subsemigroup $S$ of $T_n$ is said to be $S_n$-normal if $G_S = S_n$. In this case all the automorphisms of $S$ are inner, and $\text{Aut} S = \text{Inn} S \cong S_n$ [6].

The main result of this paper asserts that if $G_S$ contains the alternating group $\text{Alt}_n$ then $G_S = S_n$, so that $S$ is an $S_n$-normal semigroup, and $\text{Aut} S = \text{Inn} S \cong S_n$. Therefore, there is no $S \subseteq T_n$ such that $G_S = \text{Alt}_n$.

We generally use letters $h, p, g$ to denote permutations of $X_n$, and $\alpha, \beta, \gamma, \delta$ to denote non-permutations in $T_n$. In the following series of results we prove the theorem stated below.

Theorem. Let $S$ be a subsemigroup of $T_n$, $n \geq 3$. If the group $\text{Inn} S$ contains a subgroup $G$ isomorphic to $\text{Alt}_n$ then $\text{Aut} S = \text{Inn} S \cong S_n$, and $S$ is an $S_n$-normal semigroup.

Given $\alpha \in T_n$ and a subgroup $G$ of $S_n$, let $\langle \alpha : G \rangle = \langle \{hx^{-1} : h \in G\} \rangle$ be the subsemigroup of $T_n$ generated by all the conjugates of $\alpha$ by the elements of $G$. Observe that if $\beta \in \langle \alpha : G \rangle$, then $\beta = h_1 \alpha h_1^{-1} h_2 \alpha h_2^{-1} \ldots h_k \alpha h_k^{-1}$ for some $h_1, h_2, \ldots, h_k \in G$, and so for any $h \in G$, $h \beta h^{-1} = h_1 \alpha h_1^{-1} h^{-1} h_2 \alpha h_2^{-1} h^{-1} \ldots h_k \alpha h_k h^{-1} = (hh_1) \alpha (hh_1)^{-1} (hh_2) \alpha (hh_2)^{-1} \ldots (hh_k) \alpha (hh_k)^{-1} \in \langle \alpha : G \rangle$. Therefore $\langle \beta : G \rangle \subseteq \langle \alpha : G \rangle$. 

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Lemma 1. Let $G_1 \leq G_2 \leq S_n$ and $[G_2:G_1] = 2$. Let $\alpha \in T_n - S_n$. Then $\langle \alpha : G_1 \rangle = \langle \alpha : G_2 \rangle$ if and only if there is an $h \in G_2 - G_1$ such that $h \alpha h^{-1} \in \langle \alpha : G_1 \rangle$.

Proof. If $\langle \alpha : G_1 \rangle = \langle \alpha : G_2 \rangle$ then for any $h \in G_2$, $h \alpha h^{-1} \in \langle \alpha : G_1 \rangle$. To show the converse assume that $h \in G_2 - G_1$ is such that $\beta = h \alpha h^{-1} \in \langle \alpha : G_1 \rangle$. Let $p \in G_2 - G_1$. It suffices to show that $p \alpha p^{-1} \in \langle \alpha : G_1 \rangle$. Since $h, p \in G_2 - G_1$ and $[G_2:G_1] = 2$, we have $G_1 h = G_1 p$, so there exists $q \in G_1$, with $q = p h^{-1}$. Therefore $p \alpha p^{-1} = q h \alpha (q h)^{-1} = q h a h^{-1} q^{-1} \in \langle \beta : G_1 \rangle \subseteq \langle \alpha : G_1 \rangle$, as required.

The following is used to show that if $\alpha \in T_n - S_n$ then $\langle \alpha : \text{Alt}_n \rangle = \langle \alpha : S_n \rangle$.

Corollary 2. $\langle \alpha : \text{Alt}_n \rangle = \langle \alpha : S_n \rangle$ if and only if there exists an odd permutation $h$ of $X_n$ such that $h \alpha h^{-1} \in \langle \alpha : \text{Alt}_n \rangle$.

Recall that a subgroup $G$ of $S_n$ is said to be $k$-transitive if for any two $k$-subsets $A$ and $B$ of $X_n$ and any bijection $t$ from $A$ onto $B$, there exists $h \in G$ such that $h(a) = t(a)$ for every $a \in A$. We say that a subgroup $G$ of $S_n$ is $k$-block-transitive if for any two $k$-subsets $A$ and $B$ of $X_n$ there exists $h \in G$ such that $h(A) = B$. Thus any $k$-transitive semigroup is at least $k$-block-transitive. For example, $\text{Alt}_n$ is $(n-2)$-transitive [4, 10.4.6], and for all $1 \leq k \leq n-1$, $\text{Alt}_n$ is $k$-block transitive.

Given a transformation $\alpha$ of $X_n$ denote by $\pi(\alpha)$ the partition of $X_n$ determined by $\alpha$ such that $a$ and $b$ are in the same class of $\pi(\alpha)$ if and only if $\alpha(a) = \alpha(b)$. Let $\text{im} \alpha = \alpha(X_n)$ be the image of $\alpha$. Note that if $h \in S_n$ then $\pi(h \alpha h^{-1}) = \pi(h \alpha) = \{h(A) : A \in \pi(\alpha)\}$, and $\text{im}(h \alpha h^{-1}) = h(\text{im} \alpha)$.

Lemma 3. Let $G \leq S_n$ be a $k$-block transitive group. Then for any $\alpha \in T_n - S_n$ with $|\text{im} \alpha| = k$, $\langle \alpha : G \rangle$ contains an idempotent $\beta$ with $\pi(\beta) = \pi(\alpha)$.

Proof. Let $\alpha_1 (= \alpha), \alpha_2, \alpha_3, \ldots$ be conjugates of $\alpha$ by elements of $G$ such that $\text{im} \alpha_i$ is a transversal of $\pi(\alpha_{i+1})$ ($k$-block transitivity of $G$ insures their existence). Consider all the products of the form $\alpha_1, \alpha_2 \alpha_1, \alpha_3 \alpha_2 \alpha_1, \ldots$. Since $\langle \alpha : G \rangle$ is finite there exist integers $m < j$ such that $\alpha_j \alpha_{j-1} \ldots \alpha_{m+1} \alpha_m \ldots \alpha_1 = \alpha_m \ldots \alpha_1$. Let $\delta = \alpha_j \ldots \alpha_{m+1}$ and $\gamma = \alpha_m \ldots \alpha_1$. Then $\delta = \gamma$ so $\text{im} \delta \supseteq \text{im} \gamma$, and since $|\text{im} \delta| = |\text{im} \alpha| = |\text{im} \gamma|$ we have that $\text{im} \delta = \text{im} \gamma$. Thus $\delta$ is the identity on its image, and so $\delta$ is an idempotent having $\text{im} \delta = \text{im} \alpha_j$ and $\pi(\delta) = \pi(\alpha_{m+1})$. Let $h \in G$ be such that $\alpha_{m+1} = h \alpha h^{-1}$, then $\beta = h^{-1} \delta h$ is the required idempotent. Indeed $\beta^2 = h^{-1} \delta h h^{-1} \delta h = h^{-1} \delta^2 h = h^{-1} \delta h = \beta$ and $\pi(\beta) = \pi(h^{-1} \delta h) = h^{-1} (\pi(\delta)) = h^{-1} (\pi(\alpha_{m+1})) = h^{-1} (\pi(h \alpha h^{-1})) = h^{-1} (\pi(h(\alpha))) = \pi(\alpha)$.

Since $\text{Alt}_n$ is $k$-block-transitive for any $1 \leq k \leq n-1$ we have the following.

Corollary 4. $\langle \alpha : \text{Alt}_n \rangle$ contains an idempotent $\beta$ with $\pi(\beta) = \pi(\alpha)$.

We say that $\alpha$ has a partition of type $1^{k_1}2^{k_2} \ldots k_r$ if $\pi(\alpha)$ has $k_i$ classes of size $i$, where $1^{k_1}2^{k_2} \ldots k_r$ is a partition of $n$.
Let $\alpha \in T_n - S_n$ be an idempotent. There exists an $h \in S_n - \text{Alt}_n$ such that $h\alpha h^{-1} \in \langle \alpha: \text{Alt}_n \rangle$, $n \geq 3$.

**Proof.** Assume that there exist $x, y \in \alpha$ such that $\alpha^{-1}(x) = \{x\}$ and $\alpha^{-1}(y) = \{y\}$. Then for the transposition $h = (x, y)$ we have $h\alpha h^{-1} = \alpha \in \langle \alpha: \text{Alt}_n \rangle$. Now suppose $\pi(\alpha)$ contains a class $A$ having $|A| \geq 3$. Let $a, b \in A - \text{im} \alpha$. Then for $h = (a, b)$ we have $h\alpha h^{-1} = \alpha \in \langle \alpha: \text{Alt}_n \rangle$.

If none of the above holds then $\alpha$ has a partition of type $1^k 2^k = 2^k$ ($k = n/2$, $n$ is even) or $1^{k+1} 2^k$ ($k = (n-1)/2$, $n$ is odd). Let $\alpha_1, \alpha_2$ be idempotents in $T_{2k}$ and $T_{2k+1}$ respectively, $\alpha_1 = [1, 1, 3, 3, \ldots, 2k-1, 2k-1]$ and $\alpha_2 = [1, 1, 3, 3, \ldots, 2k-1, 2k-1, 2k+1]$ (we write $[a_1, a_2, \ldots, a_l]$ for a transformation mapping $i$ to $a_i$). We may assume without loss of generality that $\alpha$ equals to either $\alpha_1$ or $\alpha_2$. It is easy to verify that for $h = (12)$ and $n \geq 5$ we have

$$h \alpha_1 h^{-1} = (12)(3)(13)(12) \alpha_1 \in \langle \alpha: \text{Alt}_n \rangle.$$

If $n = 4$, then $\alpha_1 = [1, 1, 3, 3]$, and for $h = (12)$,

$$h \alpha_1 h^{-1} = ((132)(1)(123))((134)(1)(143)) \alpha_1 \in \langle \alpha: \text{Alt}_4 \rangle.$$

If $n = 3$, $\alpha_2 = [1, 1, 3]$, and for $h = (12)$,

$$h \alpha_2 h^{-1} = ((132)(132)(123)(1)(123)) \alpha_2 \in \langle \alpha: \text{Alt}_4 \rangle.$$

**Proposition 6.** Let $\alpha \in T_n$, $n \geq 3$. Then $\langle \alpha: S_n \rangle$.

**Proof.** Observe that we only need to show that $\langle \alpha: S_n \rangle \subseteq \langle \alpha: \text{Alt}_n \rangle$. If $\alpha \in \text{Alt}_n$ then $\langle \alpha: S_n \rangle \subseteq \text{Alt}_n \subseteq S_n$. Also $\langle \alpha: \text{Alt}_n \rangle \subseteq \text{Alt}_n$, and since $\text{Alt}_n$ is simple for $n \neq 4$ [4, 10.8.7] we have that $\langle \alpha: \text{Alt}_n \rangle = \text{Alt}_n$ if $\alpha \neq (1)$ and $\langle (1): \text{Alt}_n \rangle = \{(1)\}$ (provided $n \neq 4$). If $n = 4$, $\alpha \neq (1)$ and $\langle \alpha: \text{Alt}_4 \rangle \neq \text{Alt}_4$ then $\langle \alpha: \text{Alt}_4 \rangle = \mathbb{V}$, the 4-group, and $\alpha \in \mathbb{V}$. Since $\mathbb{V} \supseteq S_4$, $\langle \alpha: S_4 \rangle \subseteq \mathbb{V}$ also, and therefore $\langle \alpha: S_n \rangle \subseteq \langle \alpha: \text{Alt}_n \rangle$, as required.

If $\alpha$ is an odd permutation then for any $q \in S_n - \text{Alt}_n$, $q = \alpha(\alpha^{-1} q)$, $\alpha^{-1} q \in \text{Alt}_n$, and $q \alpha q^{-1} = \alpha(\alpha^{-1} q)(\alpha^{-1} q)^{-1} \alpha^{-1} \in \langle \alpha: \text{Alt}_n \rangle$, so $\langle \alpha: S_n \rangle \subseteq \langle \alpha: \text{Alt}_n \rangle$ again.

Now let $\alpha \in T_n - S_n$. By Corollary 4, $\langle \alpha: \text{Alt}_n \rangle$ contains an idempotent $\beta$ with $\pi(\beta) = \pi(\alpha)$. By Lemma 5 and Corollary 2, $\langle \alpha: \text{Alt}_n \rangle \supseteq \langle \beta: \text{Alt}_n \rangle = \langle \beta: S_n \rangle = \langle \alpha: S_n \rangle$ (for a transformation $\gamma$ the semigroup $\langle \gamma: S_n \rangle$ comprises all $\delta \in T_n$ having $\pi(\delta) \supseteq Q$, a partition of the same type as $\pi(\gamma)$ [2]).

**Corollary 7.** There is no $S \subseteq T_n$ such that $G_S = \text{Alt}_n$.

**Proof.** Suppose $\text{Alt}_n \subseteq G_S$. Then by Proposition 6, for any $\alpha \in S$, $h \in S_n$, we have that $h \alpha h^{-1} \in \langle \alpha: S_n \rangle = \langle \alpha: \text{Alt}_n \rangle \subseteq S$, that is $h \in G_S$ and $G_S = S_n$. 

\[ \square \]
Now to prove our main Theorem suppose that $G \leq \text{Inn} S$ such that $G \cong \text{Alt}_n$. Let $G = \{ h \in S_n : \phi_h \in G \}$. Then $G \leq G_S \leq S_n$, and the order of $G$ is at least that of $\text{Alt}_n$. Therefore $G$ contains $\text{Alt}_n$, and by Corollary 7, $G_S = S_n$.

We note that the above result is not necessarily true for semigroups of transformations of infinite sets. For example, let $X = \mathbb{Z}$ be the set of all integers and $\alpha : \mathbb{Z} \to \mathbb{Z}$ such that

$$\alpha(a) = 2a,$$

for all $a \in \mathbb{Z}$. Let $S_\mathbb{Z}$ be the symmetric group on $\mathbb{Z}$. The alternating subgroup $\text{Alt}_\mathbb{Z}$ of $S_\mathbb{Z}$ consists of all the finite even permutations of $\mathbb{Z}$. Then

$$\langle \alpha : S_\mathbb{Z} \rangle = \{ \beta : \mathbb{Z} \to \mathbb{Z} \mid \beta \text{ is } 1 - 1 \text{ and } |\mathbb{Z} - \text{im } \beta| = \aleph_0 \}$$

[3]. In particular $\langle \alpha : S_\mathbb{Z} \rangle$ contains $\beta$ defined by $\beta(a) = 2a - 1$ for all $a \in \mathbb{Z}$. Observe that for all $a \in \mathbb{Z}$, $\alpha(a) \neq \beta(a)$. Since any $h \in \text{Alt}_\mathbb{Z}$ moves at most a finite number of points, $\beta \notin \langle \alpha : \text{Alt}_\mathbb{Z} \rangle$.

For a transformation $\alpha$ of $X$ let shift $\alpha = |\{ x \in X : \alpha(x) \neq x \}|$. Let $v$ be an infinite cardinal not exceeding $|X|^+$, the cardinal successor of $|X|$, and let $\text{Sym}(X,v)$ be the subgroup of all permutations in $S_X$ whose shift is less than $v$.

**Conjecture 1.** If shift $\alpha = u$ then $\langle \alpha : \text{Sym}(X,w) \rangle = \langle \alpha : S_X \rangle$ for all $w \geq u^+$.

2. There is no semigroup $S$ of transformations of $X$ having $G_S = \text{Sym}(X,|X|)$.

Observe that permutations $h$ and $p$ in $G_S$ give rise to equal automorphisms $\phi_h$ and $\phi_p$ if and only if $h^{-1}p$ is in the centralizer $C(S)$ of $S$, $C(S) = \{ \alpha \in T_n : \alpha \beta = \beta \alpha \text{ for all } \beta \in S \}$. Thus $G_S$ is isomorphic to the group $\text{Inn} S$ of the inner automorphisms of $S$ if and only if $C(S) \cap G_S$ consists of the identity permutation. The results of this paper in conjunction with the above observations give rise to the following.

**Problem 1.** Characterize these subgroups $G$ of $S_n$ having $G = G_S$ for some subsemigroup $S$ of $S_n$.

2. Given that $G = G_T$ for some $T \subseteq T_n$ characterize all $S \subseteq T_n$ such that $G_S = G$.

3. Characterize these subsemigroups $S$ of $T_n$ having $|C(S) \cap G_S| = 1$.

**REFERENCES**


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