EFFECTIVE ASYMPTOTICS FOR SOME NONLINEAR RECURRENCES AND ALMOST DOUBLY-EXPONENTIAL SEQUENCES

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Abstract. We develop a technique to compute asymptotic expansions for recurrent sequences of the form $a_{n+1} = f(a_n)$, where $f(x) = x - ax^\alpha + bx^\beta + o(x^\beta)$ as $x \to 0$, for some real numbers $\alpha, \beta, a$, and $b$ satisfying $a > 0$, $1 < \alpha < \beta$. We prove a result which summarizes the present stage of our investigation, generalizing the expansions in [Amer. Math Monthly, Problem E 3034[1984, 58], Solution [1986, 739]]. One can apply our technique, for instance, to obtain the formula:

$$a_n = \sqrt{3} \sqrt{n} - \frac{3\sqrt{3} \ln n}{10 n^{1/2}} + \frac{9\sqrt{3} \ln n}{50 n^{3/2}} + o \left( \frac{\ln n}{n^{3/2}} \right),$$

where $a_{n+1} = \sin(a_n)$, $a_1 \in \mathbb{R}$. Moreover, we consider the recurrences $a_{n+1} = a_n^2 + g_n$, and we prove that under some technical assumptions, $a_n$ is almost doubly-exponential, namely $a_n = \lfloor k2^n \rfloor$, $a_n = \lfloor k2^n + 1 \rfloor$, $a_n = \lfloor k2^n - \frac{1}{2} \rfloor$, or $a_n = \lfloor k2^n + \frac{1}{2} \rfloor$ for some real number $k$, generalizing a result of Aho and Sloane [Fibonacci Quart. 11 (1973), 429–437].

1. Introduction

Obtaining an exact formula for the terms of a sequence given by a recurrence may not, in general, be possible. It is the intent of this paper to investigate and give asymptotics for sequences given by recurrences of the form $a_{n+1} = f(a_n)$, where $f(x) = x - ax^\alpha + bx^\beta + o(x^\beta)$ as $x \to 0$, for some real numbers $\alpha, \beta, a$, and $b$ satisfying $a > 0$, $1 < \alpha < \beta$. We also consider the same recurrence where $f(x) = x - x^2$ and give more detailed asymptotics. Moreover, we prove a few results concerning almost doubly-exponential sequences $a_{n+1} = a_n^2 + g_n$, where $-a_n + 1 < g_n < 2a_n$, generalizing a result of Aho and Sloane [1]. For standard notations consult [3], or any other book on differential and integral calculus.

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2. Asymptotics of Nonlinear Recurrences

The first part of the next lemma is known as Cesàro’s lemma, and the second part is just a small variation of the first. For completeness, we include a proof of the second part of this lemma.

**Lemma 1 (Cesàro).** Let \( \{u_n\}_{n \in \mathbb{N}}, \{v_n\}_{n \in \mathbb{N}} \) two sequences of real numbers satisfying one of the following conditions:

(i) \( \{v_n\}_{n \in \mathbb{N}} \) is eventually a strictly increasing sequence converging to infinity, or

(ii) \( \{v_n\}_{n \in \mathbb{N}} \) is eventually a strictly decreasing sequence converging to zero, and \( u_n \) converges to zero.

If the limit of the sequence \( \frac{u_{n+1} - u_n}{v_{n+1} - v_n} \) exists, then the limit of the sequence \( \frac{u_n}{v_n} \) exists, and we have the equality

\[
\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{u_{n+1} - u_n}{v_{n+1} - v_n}.
\]

**Proof.** Suppose we are given an \( \epsilon > 0 \), and by our hypothesis, for some integer \( n_0 \) and some real number \( l \) we have

\[
\left| \frac{u_{n+1} - u_n}{v_{n+1} - v_n} - l \right| < \epsilon, \quad n \geq n_0.
\]

Using (ii), the above inequality can be equivalently written in the form

\[
-\epsilon(v_n - v_{n+1}) < u_n - u_{n+1} - l(v_n - v_{n+1}) < \epsilon(v_n - v_{n+1}), \quad n \geq n_0.
\]

Adding up these inequalities from \( n \geq n_0 \) to some larger integer \( m > n \geq n_0 \), we get

\[
-\epsilon(v_n - v_m) < u_n - u_m - l(v_n - v_m) < \epsilon(v_n - v_m), \quad m > n \geq n_0.
\]

Letting \( m \) go to infinity in the above inequality and taking into account that \( u_m \to 0 \) and \( v_m \to 0 \), we obtain

\[
-\epsilon v_n \leq u_n - lv_n \leq \epsilon v_n, \quad n \geq n_0,
\]

which gives finally, after dividing by \( v_n \), the conclusion of our lemma. \( \square \)

**Theorem 2.** Suppose \( f \) is a real-valued continuous function defined on the interval \( I = (0, \delta) \) (for some \( \delta \)), which has the form \( f(x) = x - ax^\alpha + bx^\beta + o(x^\beta) \) as \( x \to 0 \), for some real numbers \( \alpha, \beta, a, \) and \( b \) satisfying \( a > 0, \ 1 < \alpha < \beta \). Then, for \( a_0 \) sufficiently small, the orbit sequence \( a_n = f(a_{n-1}) \), satisfies one of the following:

(i) if \( \beta = 2\alpha - 1 \), then

\[
a_n = \frac{1}{[a(\alpha - 1)]^{\frac{\alpha}{\alpha - 1}}} \left( \frac{1}{2} \right)^{1/(\alpha - 1)} + \frac{b - \frac{a^2}{2}}{[a(\alpha - 1)]^{\frac{\alpha}{\alpha - 1}}} \ln n + o \left( \frac{\ln n}{n^{\alpha/(\alpha - 1)}} \right),
\]
(ii) if $\beta > 2\alpha - 1$, then
\[
a_n = \frac{1}{[a(\alpha - 1)]^{n+1}} \left( \frac{1}{n} \right)^{1/(\alpha-1)} - \frac{\alpha^2}{2} \frac{\ln n}{[a(\alpha - 1)]^{n+1}} + o \left( \frac{\ln n}{n^{\alpha/(\alpha-1)}} \right).
\]

(iii) if $\beta < 2\alpha - 1$ and $b \neq 0$, then
\[
a_n = \frac{1}{[a(\alpha - 1)]^{n+1}} \left( \frac{1}{n} \right)^{1/(\alpha-1)} + b \left( \frac{a(\alpha - 1)}{a(2\alpha - 1 - \beta)} \right)^{\alpha-1} \frac{1}{n} + o \left( \frac{1}{n} \right)^{\beta-1}.
\]

Proof. We give the idea of the proof only in the case (i). Since $f(x) < x$, for $x$ in a small neighborhood of zero, the sequence $u_n$ is decreasing to zero if we assume also that $a_0$ is positive. Then we apply Cesàro’s lemma for the sequences $u_n = \frac{1}{a_n}$, and $v_n = n$:
\[
\lim_n \frac{1}{n} a_n^{\alpha-1} = \lim_n \left( \frac{1}{a_{n+1}^{\alpha-1}} - \frac{1}{a_n^{\alpha-1}} \right) = \lim_n \left( \frac{1}{f(a_n)^{\alpha-1}} - \frac{1}{a_n^{\alpha-1}} \right).
\]

Using the well-known formula from calculus $\lim_{x \to 0} \frac{1 - (1 - x)^\gamma}{x} = \gamma$, we obtain
\[
\lim_n \frac{1}{n} a_n^{\alpha-1} = \lim_n \frac{1}{a_n^{\alpha-1}} \left( 1 - \frac{1 - aa_n^{\alpha-1} + ba_n^{\beta-1} + o(a_n^{\beta-1})}{1 - aa_n^{\alpha-1} + ba_n^{\beta-1} + o(a_n^{\beta-1})} \right)^{\alpha-1}
\]
\[
= \lim_n \frac{1}{a_n} - \frac{1 - aa_n^{\alpha-1} + ba_n^{\beta-1} + o(a_n^{\beta-1})}{aa_n^{\alpha-1} - ba_n^{\beta-1} - o(a_n^{\beta-1})}
\]
\[
= (\alpha - 1)a.
\]

Equivalently, this means that $a_n = \frac{1}{[a(\alpha - 1)]^{n+1}} \left( \frac{1}{n} \right)^{1/(\alpha-1)} + o \left( \frac{1}{n} \right)^{1/(\alpha-1)}$, which is the first approximation in the statements (i)–(iii). Now let us assume that $\beta = 2\alpha - 1$. To simplify the computations we will denote $c = a(\alpha - 1)$, and $y_n = aa_n^{\alpha-1} - ba_n^{\beta-1} - o(a_n^{\beta-1})$, which under the above assumption becomes $y_n = aa_n^{\alpha-1} - ba_n^{\beta-1} - o(a_n^{\beta-1})$. We want to apply Cesàro’s lemma again for $u_n = c/n - a_n^{\alpha-1}$ and $v_n = n$:
\[
\lim_n \frac{cn - a_n^{\alpha-1}}{\ln n} = \lim_n \frac{c - \frac{1}{a_n^{\alpha-1}} + \frac{1}{a_n^{\alpha-1}}}{\ln(1 + \frac{1}{n})}
\]
\[
= \lim_n \frac{(1 - y_n)^{\alpha-1} + aa_n^{\alpha-1}(1 - y_n)^{\alpha-1} - 1}{aa_n^{\alpha-1}(1 - y_n)^{\alpha-1}}
\]
\[
= c\lim_n n^2 y_n^2 (1 - y_n)^{\alpha-1} - 1 + (\alpha - 1)y_n + n^2 (ca_n^{\alpha-1}(1 - y_n)^{\alpha-1} - (\alpha - 1)y_n).
\]
Taking into account that \( \lim_{n \to \infty} n y_n = \frac{a}{c} \) and \( \lim_{y \to 0} \frac{(1-y)^{\gamma-1} + y}{y^2} = \frac{\gamma(\gamma-1)}{2} \), we may continue the above computation as follows:

\[
\lim_n \frac{c n - 1}{a_n \ln n} = \frac{a(\alpha - 2)}{2} + c \lim_n (\alpha - 1) n^2 [a a_n^{a-1} (1 - y_n)^{a-1}]
- a a_n^{a-1} + b a_n^{2(\alpha-1)} + o(a_n^{2(\alpha-1)})] = \frac{a(\alpha - 2)}{2} + \frac{b}{a}
+ a(\alpha - 1) \lim_n n^2 a_n^{a-1} ((1 - y_n)^{a-1} - 1)
= \frac{b - a^2 a}{a}
\]

This finally says that

\[
\lim_n \frac{[c n(1/(\alpha-1)) a_n - 1]}{\ln n} = \frac{b - a^2 a}{c^2},
\]

from which (i) can be easily derived. The rest of the cases are treated similarly. □

In Odlyzko’s excellent paper [5], a few methods are studied for approximating nonlinear recurrences by linear ones. If \( f(x) = x - x^2 \), the following method for determining an approximation of \( a_n \) is presented. Let \( x_n = 1/a_n \). By iteration we obtain (cf. [2])

\[
x_n = x_{n-1} + 1 + \frac{a_{n-1}}{1 - a_{n-1}} = \cdots = \frac{1}{a_0} + n + \sum_{j=0}^{n-1} \frac{a_j}{1 - a_j}.
\]

If \( 0 < a_0 < 1 \), then we get that \( n \leq x_n \leq n + O(\log n) \), therefore \( x_n = n + \log n + o(\log n) \). In our next theorem, we push further the technique (by a somewhat similar method). We would like to mention that the function of which orbit is studied here constitutes an important case of an one-dimensional dynamical system (see Theorem 10.1, Chap. II of [4]).

**Theorem 3.** Assume \( a_{n+1} = f(a_n) \), where \( f(x) = x - x^2 \). For each \( a_1 \in I = (0, 1) \), the function \( g \) defined by

\[
g(a_1) = \lim_{n \to \infty} \left( \frac{1}{a_n} - n - \ln n \right),
\]

has the properties:

(i) \( g \) is continuously differentiable on \( I \), and for all \( x \in I \) we have \( g(x) = g(1 - x) \), and \( g(f(x)) = g(x) + 1 \);

(ii) \( g \) is strictly decreasing on \( (0, 1/2) \), strictly increasing on \( (1/2, 1) \), and its minimum value \( g(1/2) \) is a positive number;

(iii) the measure \( d\xi(x) = g'(x) dx \) is invariant under the action of \( f \) on \( (0, 1/2) \), i.e., for any measurable subset \( A \) of \((0, 1/2) \) we have \( \xi(A) = \xi(f(A)) \);
(iv) if we denote \( G_k(a_1) = \sum_{n \geq 1} \left( \frac{a_n}{1 - a_n} \right)^k \), \( k \geq 2 \), then for \( x \in (0, 1/2) \)

\[
g(x) = \ln \left( C + \int_x^{1/2} \frac{1}{t} \exp \left( \frac{1}{t} - 1 - \sum_{k=2}^{\infty} \frac{1}{k} G_k(t) \right) dt \right),
\]

where \( C = \exp(g(1/2)) \) is a constant approximately equal to 2.15768....

(v) the following expansions hold:

\[
a_n = \frac{1}{n} - \frac{\ln n}{n^2} - \frac{g(a_0)}{n^3} + \frac{(\ln n)^2}{n^3} + \frac{2(g(a_0) - 1) \ln n}{n^3} + o \left( \frac{\ln n}{n^3} \right),
\]

\[
\frac{1}{a_n} = n + \ln n + g(x) + \frac{\ln n}{n} + \frac{(-\frac{1}{2} + g) \ln n}{n^2} - \frac{1}{2} \frac{(\ln n)^2}{n^2}
\]

\[
+ \frac{3}{2} \frac{(3 - g) \ln n}{n^2} + \frac{3}{2} \frac{g - 1}{2} \frac{g^2 - 5}{6} \frac{1}{n^2} + \frac{1}{3} \frac{(\ln n)^3}{n^3}
\]

\[
+ (-2 + g) \frac{(\ln n)^2}{n^3} + \left( \frac{19}{6} - 4g + g^2 \right) \frac{\ln n}{n^3} + o \left( \frac{\ln n}{n^3} \right).
\]

Proof. The sequence \( x_n = \frac{1}{a_n} \), \( n \geq 1 \), satisfies the recurrence relation \( x_{n+1} = h(x_n) \), where \( h(x) = x + 1 + \frac{1}{x - 1} \), for \( x \in (1, \infty) \). If we define \( r(x_1) = \lim_{n \to \infty} y_n \) with \( y_n = x_n - n - \ln n \), clearly \( g(x) = r(1/x) \) for all \( x \in I \). Since all the properties of \( r \) transfer to \( g \) in a corresponding way, we prefer to work with the function \( r \) instead of \( g \). Directly from the recurrence relation for \( x_n \) we easily see that \( x_n \) is a strictly increasing sequence, \( x_2 \geq 4, (h(1, \infty)) = [4, \infty) \), and we get

\[
x_{n+1} = x_2 + n - 1 + \sum_{k=2}^{n} \frac{1}{x_k - 1}, \quad n \geq 2.
\]

From this we obtain that \( x_n \geq n + 2 \) for all \( n \geq 2 \). This shows, in particular, that the limit defining \( r \) exists, since \( y_n \) is a decreasing sequence:

\[
y_n - y_{n+1} = \ln(1 + \frac{1}{n}) - \frac{1}{x_{n-1}} - \frac{1}{x_{n+1}} > \frac{1}{n + 1} - \frac{1}{x_{n+1}} \geq 0, \quad n \geq 2.
\]

Secondly, going back to (5), the next better estimation from above of \( x_n \) results:

\[
x_{n+1} \leq x_2 + n - 1 + \sum_{k=2}^{n} \frac{1}{k+1} < x_2 + n - 1 + \ln(n+1) - \ln 2, \quad n \geq 2.
\]

Since for \( u > v \geq 2 \) or \( 1 < u < v \leq 2 \), we get \( h(u) > h(v) \geq 4 \), and then \( h(h(u)) > h(h(v)) \geq 4 \), a simple induction argument shows that \( r \) is decreasing on \([1, 2]\) and increasing on \([2, \infty)\). Therefore, in order to prove that \( r \) has finite values, it is enough to show that \( r(2) > 0 \). Hence, if \( x_1 = 2 \), (6) becomes

\[
x_n \leq n + \omega + \ln n, \quad n \geq 2,
\]

where \( \omega = 2 - \ln 2 > 1 \). Using (7) in (5), we obtain

\[
x_{n+1} \geq n + 3 + \sum_{k=2}^{n} \frac{1}{k-1 + \omega + \ln k}, \quad n \geq 2.
\]
This implies that for \( n \geq 2 \)
\[
y_{n+1} \geq 2 - \ln(n+1) + \sum_{k=1}^{n-1} \frac{1}{k + \omega + \ln(k+1)}
> 2 - \ln(n+1) + \int_1^n \frac{dx}{x + \omega + \ln(x+1)}.
\]
Since \( \frac{1}{x + \omega + \ln(x+1)} > \frac{1}{(x + \omega)} - \ln(x+1) \) on the interval \([1, \infty)\), we can continue the above sequence of inequalities as follows:
\[
y_{n+1} \geq 2 - \ln(n+1) + \int_1^n \frac{dx}{x + \omega} - \int_1^n \frac{\ln(x+1)dx}{(x + \omega)^2}
= 2 - \ln(1 + \omega) + \ln(n + \omega) - \int_1^n \frac{\ln(x+1)dx}{(x + \omega)^2}
> 2 - \ln(1 + \omega) - \int_1^\infty \frac{\ln(x+1)dx}{(x + \omega)^2}
= 2 + \frac{2 \ln 2}{\omega - 1} - \frac{\omega}{\omega - 1} \ln(1 + \omega).
\]
Since \( \ln(1 + \omega) = \ln 2(1 + \frac{\omega - 1}{2}) < \ln 2 + \frac{\omega - 1}{2} = \frac{3 - \omega}{2} \), we obtain from the above computation that
\[
r(2) = \lim_n y_{n+1}(2) \geq \frac{(\omega - 1)(\omega + 4)}{2(\omega + 1)} > 0.
\]
Hence we have proved the second part of the statement (ii) in Theorem 3.

We next look at the sequence of the derivatives of the functions \( x_n(x) = h^n(x)(x_1 = x) \), where \( h^{n+1}(x) = h(h^n(x)) \), \( n \geq 1 \). Since \( h'(x) = 1 - \frac{1}{(x-1)^2} \), and \( h^n''(x) = h'(h^{n-1}(x))h'(h^{n-2}(x)) \ldots h'(x) \), we get
\[
y'_n = x'_n = \prod_{k=1}^{n-1} \left(1 - \frac{1}{(x_k - 1)^2}\right), \quad n \geq 2.
\]
Using the inequality \( x_n \geq n + 2, n \geq 2 \), the product appearing in (8) is absolutely convergent. Therefore the sequence \( y_n(x) = y_n(2) + \int_2^x y'_n(t)dt \) converges to
\[
r(x) = r(2) + \int_2^x \prod_{k=1}^{\infty} \left(1 - \frac{1}{(x_k(t) - 1)^2}\right)dt.\]
In particular, this shows that \( r \) is continuously differentiable. In order to complete the proof of (i), let us observe that
\[
r(h(x)) = \lim_n y_n(h(x)) = \lim_n x_{n+1}(x) - n - \ln n =
= \lim_n x_n(x) + 1 + \frac{1}{x_n - 1} - n - \ln n
= r(x) + 1.
\]
Hence \( g(f(x)) = r(1/f(x)) = r(h(1/x)) = r(1/x) + 1 \) and \( g(1-x) = g(f(1-x)) - 1 = g(f(x)) - 1 = g(x) \), for \( x \in I \), which completes the proof of (i). Because

\[
\sum_{k=1}^{\infty} \left( 1 - \frac{1}{(x_k(x) - 1)^2} \right) = \frac{x(x-2)}{(x-1)^2} \prod_{k=2}^{\infty} \left( 1 - \frac{1}{(x_k(x) - 1)^2} \right),
\]

(9) \( r'(x) \), it is easy to see that \( r'(x) > 0 \) for \( x > 2 \) and \( r'(x) < 0 \) for \( 1 < x < 2 \). This completes the proof of (ii).

To get (iii) we can use (i) to obtain \( g'(f(x))f'(x) = g'(x) \), and hence by the change of variable formula,

\[
\xi(f(A)) = \int_{f(A)}^{A} d\xi(x) = \int_{f(A)} g'(x) dx = \int_{f(A)} g'(f(x))f'(x) dx = \int_{f(A)} g'(f(x))f'(x) dx = \int_{A} g'(x) dx = \int_{A} d\xi(x) = \xi(A).
\]

In order to prove (iv), let us compute \( \ln(r'(x)) \) for \( x > 2 \), using formula (9) and the recursive relation:

\[
\ln(r'(x)) = \ln \left( \prod_{k=1}^{\infty} \left( 1 - \frac{1}{(x_k(x) - 1)^2} \right) \right) = \ln \left( \lim_{n} \prod_{k=1}^{n} \left( 1 - \frac{1}{x_k(x) - 1} \right) \prod_{k=1}^{n} \left( 1 + \frac{1}{x_k(x) - 1} \right) \right) = \ln \left( \sum_{k=1}^{n} \ln \left( 1 - \frac{1}{x_k(x) - 1} \right) + \ln \left( \prod_{k=1}^{n} \frac{x_k(x)}{x_k(x) - 1} \right) \right) = \ln \left( -\sum_{k=1}^{n} \sum_{j=1}^{\infty} \frac{1}{j} \left( \frac{1}{x_k(x) - 1} \right)^j + \ln \left( \prod_{k=1}^{n} \frac{x_{k+1}(x)}{x_k(x)} \right) \right).
\]

Here, we used the definition of \( \{x_k\} \), that is, \( x_{k+1} = h(x_k) = x_k + 1 + \frac{1}{x_k-1} \), therefore \( \frac{x_{k+1}}{x_k} = \frac{x_k+1}{x_k} \), hence the last equality. After we interchange the sums,
using (5) we can continue the above computation as follows:

\[
\ln(r'(x)) = \lim_{n} \left( \ln(x_{n+1}(x)) - \ln x - \sum_{k=1}^{n} \frac{1}{x_k(x)} - 1 - \sum_{j=2}^{\infty} \sum_{k=1}^{n} \frac{1}{j} \left( \frac{1}{x_k(x) - 1} \right)^j \right)
\]

\[
= -\ln x + \lim_{n} \left( \ln(x_{n+1}(x)) - x_{n+1}(x) + n + x - \sum_{j=2}^{\infty} \sum_{k=1}^{n} \frac{1}{j} \left( \frac{1}{x_k(x) - 1} \right)^j \right)
\]

\[
= x - 1 - \ln x - \lim_{n} \left( x_{n+1}(x) - (n + 1) - \ln(n + 1) + \ln \left( \frac{n + 1}{x_{n+1}(x)} \right) \right)
\]

\[
- \lim_{n} \left( \sum_{j=2}^{\infty} \sum_{k=1}^{n} \frac{1}{j} \left( \frac{1}{x_k(x) - 1} \right)^j \right).
\]

Since the double sum \( \sum_{j=2}^{\infty} \sum_{k=1}^{n} \frac{1}{j} \left( \frac{1}{x_k(x) - 1} \right)^j \) is absolutely convergent we can interchange the limit sign with the sum sign in the above computation, and using the definition of \( r \) we obtain the following differential equation in \( r \):

\[
\ln(r'(x)) = x - 1 - \ln x - r(x) - \sum_{j=2}^{\infty} \sum_{k=1}^{n} \frac{1}{j} \left( \frac{1}{x_k(x) - 1} \right)^j,
\]

or

\[
(10) \quad r'(x) \exp(r(x)) = \frac{1}{x} \exp(x - 1 - R(x)),
\]

where \( R(x) = \sum_{j=2}^{\infty} \sum_{k=1}^{n} \frac{1}{j} \left( \frac{1}{x_k(x) - 1} \right)^j \). Integrating (10), we obtain a formula which gives us another way of approximating the values of \( r \):

\[
(11) \quad r(x) = \ln \left( C + \int_{2}^{x} \frac{1}{t} \exp \left( t - 1 - R(t) \right) dt \right), \quad x > 2.
\]

In terms of the function \( g \) and the sequence \( \{a_n\} \), after a change of variable, the formula (11) becomes

\[
g(x) = \ln \left( C + \int_{x}^{1} \frac{1}{u} \exp \left( \frac{1}{u} - 1 - G(u) \right) du \right), \quad x \in (0, 1/2),
\]

where \( G(u) = R(1/u) = \sum_{j=2}^{\infty} \sum_{k=1}^{n} \frac{1}{j} \left( \frac{1}{x_k(1/u) - 1} \right)^j = \sum_{j=2}^{\infty} \sum_{k=1}^{n} \frac{1}{j} \left( \frac{a_k(u)}{1 - a_k(u)} \right)^j \),

and (iv) is proved.
To prove (v), we apply several times part (ii) of Cesàro's Lemma. First we take $u_n = x_n(x) - n - \ln n - r(x)$ and $v_n = (1/n) \ln n$:

$$\lim_n \frac{n(x_n(x) - n - \ln n - r(x))}{\ln n} = \lim_n \frac{1}{x_{n-1}^{(n+1)}} - \frac{\ln n}{n} = 1.$$  

Using the same technique we can compute the other terms in (11), and (10) is easily obtained from (11). □

We point out that there are cases when it is easy to determine expansions as in (4) for all $k \geq 2$. For example, if $f(x) = x/(1 + x)$, then $\{a_n\}$ has the expansion

$$a_n = \sum_{j=0}^{m} \frac{(-1)^j}{n^{j+1} a_0^j} + o\left(\frac{1}{n^{m+1}}\right), \quad n \geq 1, \quad a_0 \in (0, \infty).$$

That can be seen easily by linearizing the recurrence $a_{n+1} = f(a_n)$ replacing $1/a_n$ by $b_n$. We obtain the linear equation $b_{n+1} = b_n + 1$, which obviously produces $a_n = \frac{1}{n + a_0}$, from which we infer the previous approximation.

On the other hand, if $f(x) = \sin x$, we computed using Theorem 2 the following expansion:

$$a_n = \frac{\sqrt{3}}{\sqrt{n}} - \frac{3\sqrt{3}}{10} \frac{\ln n}{n^{1/2}} + \frac{9\sqrt{3}}{50} \frac{\ln n}{n^{2/2}} + o\left(\frac{\ln n}{n^{2+1/2}}\right),$$

where the coefficients do not seem to depend on the initial value of the sequence.

3. **Almost Doubly-Exponential Sequences**

Aho and Sloane [1] considered the sequences of the form $a_{n+1} = a_n^2 + g_n$, where $|g_n| \leq a_n/4$, $a_n \geq 1$ and $|\log(a_{n+1}a_n^{-2})|$ is decreasing, for $n \geq n_0$. They proved that under these conditions, there exists a constant $k$ such that $a_n = \text{nearest integer to } k^{\omega n}$. Obviously, the sequence of Theorem 3 is not among the ones considered by Aho and Sloane, since it does not satisfy the mentioned conditions. In the spirit of [1], relaxing the conditions, using a somewhat different method, we prove the next theorem, involving what we call almost doubly-exponential recurrences. We denote by $\exp(x)$ the exponential function $e^x$ with Euler's constant base.

**Theorem 4.** Let the sequence of positive integers $a_{n+1} = a_n^2 + g_n$, satisfying $-a_n + 1 < g_n < a_n$, $a_0 > 1$ and $|\log(a_{n+1}a_n^{-2})|$ is decreasing (for $n \geq n_0$). Then there exists $\alpha$ such that $a_n = [\exp(2^n\alpha)]$, or $a_n = [\exp(2^n\alpha)] + 1$ (for $n \geq n_0$).

**Proof.** Since the entire proof refers to $n \geq n_0$, we may as well assume that $n_0 = 0$. The proof uses some ideas of [1] and [5]. Let $u_n := \log a_n$, and
\[ \delta_n := \log(g_n a_n^{-2} + 1). \] Thus \( u_{n+1} = 2u_n + \delta_n. \) Iterating we get
\[ u_n = 2^n u_0 + 2^n \sum_{k=0}^{n-1} \delta_k 2^{-k-1}. \]

The series \( \alpha := u_0 + \sum_{k=0}^{\infty} \delta_k 2^{-k-1} \) is absolutely convergent since \(|\delta_k| < \log(1 + a_k^{-1}) < \log 2\). Taking \( r_n := 2^n \alpha - u_n \), we get that \( a_n = \exp(u_n) = \exp(2^n \alpha - r_n). \) Now,
\[ \exp(2^n \alpha) = a_n \exp(r_n), \quad \text{and} \]
\[ r_n = 2^n \sum_{k=n}^{\infty} \delta_k 2^{-k-1} = \sum_{k=0}^{\infty} \delta_{k+n} 2^{-k-1}. \]

Since \(|\log(a_{n+1} a_n^{-2})| = |\log(g_n a_n^{-2} + 1)| = |\delta_n|\) is decreasing, we get
\[ |r_n| \leq \sum_{k=0}^{\infty} |\delta_{k+n}| 2^{-k-1} \leq |\delta_n| \sum_{k=0}^{\infty} 2^{-k-1} = |\delta_n| \]
which implies
\[ a_n \exp(-|\delta_n|) \leq \exp(2^n \alpha) \leq a_n \exp(|\delta_n|). \]

We use now the definition of \( \delta_n \), and deduce
\[ \exp(\delta_n) = g_n a_n^{-2} + 1, \]
\[ \exp(-\delta_n) = (g_n a_n^{-2} + 1)^{-1}. \]

Therefore, using (13) and (14), if \( \delta_n > 0 \), then
\[ a_n - \exp(2^n \alpha) \leq a_n - a_n \exp(-\delta_n) = a_n (1 - (g_n a_n^{-2} + 1)^{-1}) \]
and
\[ a_n - \exp(2^n \alpha) \geq a_n - a_n \exp(\delta_n) = a_n (1 - (g_n a_n^{-2} + 1)) = -g_n a_n^{-1}. \]

Now, in (15) to have \( a_n (1 - (g_n a_n^{-2} + 1)^{-1}) < 1 \), it is necessary to have
\( (g_n a_n^{-2} + 1)^{-1} > 1 - 1/a_n \)
which in turn is equivalent to \( g_n < \frac{a_n^2}{a_n - 1} = a_n + 1 + \frac{1}{a_n - 1} \). The last inequality is true since \( g_n < a_n \). In (16) to have \(-g_n a_n^{-1} > -1\), it is necessary to have \( g_n < a_n \).

If \( \delta_n < 0 \), by (13) and (14), then
\[ a_n - \exp(2^n \alpha) \leq a_n - a_n \exp(\delta_n) = a_n (1 - (g_n a_n^{-2} + 1)) = -g_n a_n^{-1}, \]
\[ a_n - \exp(2^n \alpha) \geq a_n - a_n \exp(-\delta_n) = a_n (1 - (g_n a_n^{-2} + 1)^{-1}) \]
Now, in (17), \(-g_n a_n^{-1} < 1\) is equivalent to \( g_n > -a_n \), and the last inequality is certainly true, since \( g_n > -a_n + 1 \). In (18) to have \( a_n (1 - (g_n a_n^{-2} + 1)^{-1}) > -1 \), it is necessary to have \( g_n a_n^{-2} + 1 > \frac{a_n}{a_n + 1} = 1 - \frac{1}{a_n + 1} \). That is equivalent to
\( g_n > \frac{-a_n^2}{a_n + 1} = -a_n + 1 - \frac{1}{a_n + 1}, \) which is certainly true, as \( g_n \) is an integer, \( a_n > 1 \) and \( g_n > -a_n + 1. \)

Thus, we obtain, in any case, that \( |a_n - \exp(2^n \alpha)| < 1, \) which implies (since \( a_n \) is an integer) that \( a_n = \lfloor \exp(2^n \alpha) \rfloor, \) or \( a_n = \lfloor \exp(2^n \alpha) \rfloor + 1. \)

Remark 5. The previous theorem does not consider the case of \( g_n = -a_n + 1 \) (the lower bound). However, in that case we get \( a_{n+1} = a_n^2 - a_n + 1, \) which was dealt with by Aho and Sloane (Recurrence 2.4), if \( a_1 = 2, \) being transformed into a recurrence satisfying their conditions, deriving the solution \( |k^{2^n} + \frac{1}{2}|, \) for some real number \( k. \)

Consider now that case of \( a_n < g_n < 2a_n \) in the recurrence \( a_{n+1} = a_n^2 + g_n, \) \( a_n > 1 \) positive integers. Let \( g_n' = g_n - a_n. \) Thus, \( 0 < g_n' < a_n \) and the recurrence can be written as

\[ a_{n+1} = a_n^2 + a_n + g_n'. \]

Let \( b_n = a_n + \frac{1}{2} \) and \( h_n = g_n' - \frac{3}{4} = g_n - a_n - \frac{3}{4}. \) It follows that

\[ b_{n+1} = b_n^2 + h_n, \]

with \( -\frac{3}{4} < h_n < a_n - \frac{3}{4} < a_n, \)

which is of the first type, but (beware!) this sequence does not consist of integers. We start with one observation: since \( a_n < g_n, \) it follows that \( g_n - a_n \geq 1, \) therefore \( h_n \geq \frac{1}{4}, \) so \( h_n \) satisfies \( 0 < h_n < a_n. \)

Let \( u_n := \log b_n, \) and \( \delta_n := \log(h_n b_n^{-2} + 1). \) If \( |\log(b_{n+1} b_n^{-2})| \) is decreasing, the same technique as before renders, since \( h_n > 0, \)

\[ b_n - \exp(2^n \beta) \leq b_n(1 - (h_n b_n^{-2} + 1)^{-1}), \]
\[ b_n - \exp(2^n \beta) \geq -h_n b_n^{-1}, \]

where \( \beta := u_0 + \sum_{k=0}^{\infty} \delta_k 2^{-k-1}. \) Moreover, \( b_n(1 - (h_n b_n^{-2} + 1)^{-1}) < 1 \) if and only if

\[ \frac{b_n - 1}{b_n} < \frac{1}{h_n b_n^{-2} + 1}. \]

This is equivalent to \( h_n < \frac{b_n^2}{b_n - 1} = b_n + 1 + \frac{1}{b_n - 1}, \) which is certainly true as \( h_n < a_n < a_n + \frac{1}{2} = b_n. \) Furthermore, since \( -h_n b_n^{-1} > -1, \) then

\[ -\frac{3}{2} < a_n - \exp(2^n \beta) < \frac{1}{2}. \]

The right hand side inequality is improved by the simple observation that since \( \delta_k > 0, \) then \( 2^n \beta > u_n, \) therefore, \( \exp(2^n \beta) > b_n = a_n + \frac{1}{2}, \) which implies

\[ -\frac{3}{2} < a_n - \exp(2^n \beta) < -\frac{1}{2}, \]

and so,

\[ a_n < \exp(2^n \beta) - \frac{1}{2} < a_n + 1. \]
To cover the whole range \( -a_n + 1 < g_n < 2a_n \), it suffices to study the case of \( g_n = a_n \). In that case, we get the recurrence of positive integers \( a_{n+1} = a_n^2 + a_n \).

Taking \( b_n = a_n + 1/2 \), we get

\[
\frac{b_{n+1}}{b_n} = \frac{b_n^2 - \frac{3}{4}}{b_n} = \frac{b_n^2 - 3}{4},
\]

which was dealt with by Aho and Sloane, if \( b_1 = \frac{3}{2} \), obtaining \( b_n = \frac{3}{2} + [k2^n + \frac{3}{2}] \), \( n \geq 3 \), for some real \( k \).

Thus, we have proved

**Theorem 6.** Let the recurrence of positive integers \( a_{n+1} = a_n^2 + g_n \), where \( a_n < g_n < 2a_n \), \( a_n > 1 \) (if \( n \geq n_0 \)). Also assume that \( \left| \log \left( \frac{(a_{n+1} + 1/2)(a_n + 1/2)^{-2}}{a_n} \right) \right| \) is decreasing. Then there exists a real number \( \beta \) such that

\[
a_n = \left\lfloor \exp(2^n \beta) - \frac{1}{2} \right\rfloor, \quad \text{if } n \geq n_0.
\]

If \( a_{n+1} = a_n^2 + a_n \) and \( a_1 = 1 \), then

\[
a_n = \left\lfloor \exp(2^n \beta) + \frac{5}{2} \right\rfloor, \quad \text{if } n \geq 3.
\]

Certainly the theorem can be further extended by taking various other intervals for \( g_n \) and imposing the restrictive decreasing property on \( a_n \).

The sequence \( g_n \) may or may not depend on \( a_n \). If \( g_n = a_n - 2a_n^2 \), we end up with a recurrence of the form \( a_{n+1} = f(a_n) \), where \( f(x) = x - x^2 \). Obviously, in this case Theorem 4 is not true, since the inequality imposed on \( g_n \) does not hold. But this case was dealt with by Theorem 3.

Can we relax the conditions of Theorem 4 and Theorem 6 even further? The answer is yes, but the result is not that accurate. Let the recurrence of positive integers \( a_{n+1} = a_n^2 + h_n \) with \( |h_n| < (1 + \epsilon) a_n \), \( a_n \geq 1 \), where \( \epsilon > 0 \) is a fixed parameter. In the same manner as before, we denote by \( \delta_n(\epsilon) = \log(h_n a_n^{-2} + 1) \) and \( u_n = \log a_n \). The series \( \alpha(\epsilon) = u_0 + \sum_{k=0}^{\infty} \delta_k(\epsilon) 2^{-k-1} \) is convergent since

\[
- \log(2 - \epsilon) \leq \log(1 - \frac{1 + \epsilon}{a_k}) < \delta_k(\epsilon) < \log(1 + \frac{1 + \epsilon}{a_k}) < \log(2 + \epsilon),
\]

for \( k \) sufficiently large so that \( a_k > 1 + \epsilon \). Taking \( r_n = 2^\alpha - u_n \), we get that \( a_n = \exp(u_n) = \exp(2^n \alpha - r_n) \). We did not impose the decreasing property on \( \delta_n(\epsilon) \), so we can only infer at this stage that

\[
- \log(2 + \epsilon) \leq r_n = \sum_{k=0}^{\infty} \delta_{k+n}(\epsilon) 2^{-k-1} \leq \log(2 + \epsilon),
\]

using the double inequality on \( \delta_n(\epsilon) \).
With a bit more work, we conclude

**Proposition 7.** Let $a_{n+1} = a_n^2 + h_n$ with $|h_n| < (1 + \epsilon)a_n$, $a_n \geq 1$, where $\epsilon \geq 0$ is a fixed parameter. Then there exists a constant $\alpha$ such that

$$\frac{1}{2 + \epsilon} \exp(2^n \alpha) \leq a_n \leq (2 + \epsilon) \exp(2^n \alpha),$$

if $n$ is sufficiently large so that $a_n > 1 + \epsilon$.

**References**


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