AUTOMORPHISMS OF NORMAL PARTIAL TRANSFORMATION SEMIGROUPS

by INESSA LEVI

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1. Introduction. We let $X$ be an arbitrary infinite set. A semigroup $S$ of total or partial transformations of $X$ is called $\mathcal{G}_X$-normal if $hSh^{-1} = S$, for all $h$ in $\mathcal{G}_X$, the symmetric group on $X$. For example, the full transformation semigroup $\mathcal{I}_X$, the semigroup of all partial transformations $\mathcal{P}_X$, the semigroup of all 1–1 partial transformations $\mathcal{I}_X$ and all ideals of $\mathcal{I}_X$, $\mathcal{P}_X$ and $\mathcal{I}_X$ are $\mathcal{G}_X$-normal.

If $S$ is a $\mathcal{G}_X$-normal semigroup then for each $h \in \mathcal{G}_X$ the map

$$\phi : f \mapsto hfh^{-1} \quad (f \in S)$$

is an inner automorphism of $S$. The set Inn $S$ of all inner automorphisms of $S$ is a subgroup of the group Aut $S$ of all automorphisms of $S$. In [3] we showed that if $S$ is a $\mathcal{G}_X$-normal subsemigroup of $\mathcal{I}_X$ then inner automorphisms exhaust all automorphisms of $S$, that is

$$\text{Aut } S = \text{Inn } S.$$

The purpose of this paper is to extend the above result to an arbitrary $\mathcal{G}_X$-normal subsemigroup $S$ of $\mathcal{P}_X$ and therefore to give a complete description of all automorphisms of any $\mathcal{G}_X$-normal semigroup.

Schreier [10] in 1937 was the first to show that Aut $\mathcal{I}_X = \text{Inn } \mathcal{I}_X$. Since then many authors have described the automorphisms of various $\mathcal{G}_X$-normal semigroups: Mal'cev [5] (all ideals of $\mathcal{I}_X$); Liber [4] ($\mathcal{I}_X$ and all its ideals); Gluskin [1] ($\mathcal{P}_X$); Shutov [8] (the semigroup of all partial transformations shifting at most a finite number of elements); Shutov [9] (all ideals of $\mathcal{P}_X$); Schein [6, 7] (all $\mathcal{G}_X$-normal subsemigroups of $\mathcal{I}_X$, but see [2] for a special case). In [11] Sullivan showed that if $S$ is a subsemigroup of $\mathcal{P}_X$ containing a constant idempotent with the range $\{x\}$, for each $x \in X$, then Aut $S = \text{Inn } S$. In particular if $S$ is a $\mathcal{G}_X$-normal subsemigroup of $\mathcal{P}_X$ containing a constant map then Aut $S = \text{Inn } S$. Our result completes the task of characterization of all automorphisms of a $\mathcal{G}_X$-normal semigroup, subsuming previously stated results for $\mathcal{G}_X$-normal semigroups.

In this paper we continue the development of a technique involving the production of certain maximal one-sided ideals, first introduced in [3]. Here the assumption (made due to [3]) that $S$ contains a proper partial transformation allows us to restrict ourselves to the study of only left ideals. Hence, unlike in [3], a uniform proof is given for the case when $S \subseteq \mathcal{I}_X$ as well as when $S$ contains transformations which are not 1–1.

2. Transitivity. We say that a semigroup $S$ is trivial if $S \subseteq \{\Phi, \iota\}$, where $\Phi$ is the empty and $\iota$ is the identity transformation. In what follows $S$ is non-trivial. The composition of transformations $f$ and $g$ in $S$ defined by the formula

$$fg(x) = f(g(x)), \quad \text{where } x \in X.$$
In this section we show that each non-trivial $\mathcal{B}_X$-normal semigroup $S$ is transitive. If $S$ also is a constant-free semigroup then it is 2-transitive (Definition 2.3).

For an $f$ in $\mathcal{P}_X$ we denote the range of $f$ by $R(f)$, the domain of $f$ by $D(f)$ and the partition of $f$ by $\pi(f)$ ($=\{f^{-1}(x):x \in R(f)\}$). If $S$ is a subsemigroup of $\mathcal{P}_X$, let

$$D(S) = \{D(f): f \in S\} \text{ and } \pi(S) = \{\pi(f): f \in S\}.$$ 

We say that $D(S)$ ($\pi(S)$) is normal if, for each $h \in \mathcal{B}_X$, $h(D(S)) = D(S)$ ($h(\pi(S)) = \pi(S)$).

The following lemma is straightforward.

**Lemma 2.1.** If $S$ is a $\mathcal{B}_X$-normal semigroup, then $D(S)$ and $\pi(S)$ are normal.

The proof of our next proposition coincides with the proof of result 1.3 of [3].

**Proposition 2.2.** Every $\mathcal{B}_X$-normal semigroup is transitive.

**Definition 2.3.** A semigroup $S$ is 2-transitive if for any two ordered subsets $\{x, u\}$ and $\{y, v\}$ of $X$ ($x \neq u, y \neq v$) there exists an $f$ in $S$ with $f(x) = y, f(u) = v$.

**Lemma 2.4.** If $S$ is a $\mathcal{B}_X$-normal constant-free semigroup then each $f$ in $S$ has an infinite range.

**Proof.** Suppose $R(f)$ is finite. Then either $D(f)$ is finite and $\exists g \in S$ with $|D(g) \cap R(f)| = 1$ (by 2.1), or $\pi(f)$ contains an infinite subset $A$ and $\exists q \in S$ with $R(f) \subseteq B \in \pi(q)$ (by 2.1). In either case $S$ contains a constant map $(gf$ or $qf)$.

**Proposition 2.5.** Every $\mathcal{B}_X$-normal constant-free semigroup $S$ is 2-transitive.

**Proof.** Take arbitrary ordered subsets $\{x, u\}$ and $\{y, v\}$ of $X$, $x \neq u, y \neq v$. We construct an $f$ in $S$ such that $f(x) = y$ and $f(u) = v$.

Firstly let $x, y, u$ and $v$ be distinct. Choose $t$ in $S$ with $t(x) = y$ (by 2.2) and let $z \in D(t) \setminus \{x, y, t^{-1}(x), t^{-1}(y)\}$ (if such $z$ does not exist then $R(t) \subseteq \{x, y, t(y)\}$, a contradiction to 2.4). Let $g = (z, u)t(z, u)$ and $g(u) = (z, u)t(z) = w$ (here $(z, u)$ denotes the permutation of $X$ interchanging $z$ and $u$ and leaving all other elements of $X$ fixed). Clearly $g(x) = y$, and if $w = v$, then $f = g$. If $w \neq v$, $u$ then let $f = (v, w)g(v, w)$ (since $z \notin \{t^{-1}(x), t^{-1}(y)\}$, $w \neq x, y$, and this ensures $f(x) = y$).

Thus starting with $t \in S$, $t(x) = y$, we construct either the required $f$ or a map $g$ with $g(x) = y, g(u) = u$. Similarly, starting with $s \in S$, $s(u) = v$, we can construct either the required $f$ or a map $q$ with $q(u) = v, q(x) = x$. In the latter case we let $f = (u, v)g(u, v)q$.

Now assume that $x, y, u$ and $v$ are not all distinct. Choose $a$ and $b$ in $X \setminus \{x, y, u, v\}$, $a \neq b$, and with the aid of the first part of the proof construct $r, s \in S$ with $r(x) = a, r(u) = b$ and $s(a) = y, s(b) = v$. Then $f = sr$ is the required map.

3. **Left ideals and automorphisms.** Let $S$ be a non-trivial $\mathcal{B}_X$-normal constant-free semigroup. If $S \subseteq \mathcal{T}_X$, then $\text{Aut} S = \text{Inn} S$ [3]. Hence we assume that $S$ contains a proper partial transformation and show that all automorphisms of $S$ are inner.
Definition 3.1. Given distinct \( f, g \in S \) let

\[ \mathcal{L}(f, g) = \{ l \in S : lf = lg \} \]

Then \( \mathcal{L}(f, g) \) is a left ideal of \( S \), which we call a function left ideal.

We will show in 3.12 that there always exist \( f, g \in S \) with \( \mathcal{L}(f, g) \neq \{ \Phi \} \). However, \( \mathcal{L}(f, g) \) may consist of the empty map. Let \( S \), for example, be the semigroup of all 1-1, onto transformations \( f \) with \( |X \setminus D(f)| = |X| \). Choose an \( f \) in \( S \). Clearly \( X \setminus D(f) \in D(S) \), and so we can choose a \( g \) in \( S \) with \( D(g) = X \setminus D(f) \). Then \( \mathcal{L}(f, g) = \{ \Phi \} \), because for any \( l \in S \), \( lf = lg \) implies

\[ D(f) \supseteq D(lf) = D(lg) \subseteq D(g) = X \setminus D(f), \]

so \( lg = \Phi \). But then \( D(l) \cap X = D(l) \cap R(g) = \Phi \), the empty set. Thus \( l = \Phi \).

If \( \phi \in \text{Aut} \ S \), then for any \( f, g \in S \)

\[ \phi(\mathcal{L}(f, g)) = \phi(\{ l \in S : lf = lg \}) = \{ l' \in S : l' \phi(f) = l' \phi(g) \} = \mathcal{L}(\phi(f), \phi(g)). \]

Similar equality holds for \( \phi^{-1} \in \text{Aut} \ S \) and we deduce the following result.

Lemma 3.2. Any \( \phi \in \text{Aut} \ S \) permutes function left ideals and \( \phi(\mathcal{L}(f, g)) = \mathcal{L}(\phi(f), \phi(g)). \)

Our aim is to translate the definition of \( \mathcal{L}(f, g) \) from the language of transformations to the language of subsets of \( X \) (Proposition 3.11), and to obtain a bijection of \( X \) associated with \( \phi \), specifically, with the permutation of function left ideals by \( \phi \).

Definition 3.3. Let \( x \in X \) and

\[ \mathcal{L}(x) = \{ l \in S : x \in X \setminus D(l) \}. \]

Then \( \mathcal{L}(x) \) is a left ideal of \( S \), which we call a point left ideal.

Notice that since \( S \) contains a proper partial transformation, 2.1 ensures that \( \mathcal{L}(x) \neq \Phi \), for any \( x \in X \).

Lemma 3.4. Given \( x, y \in X \) the following three statements are equivalent:

(i) \( \mathcal{L}(x) \subseteq \mathcal{L}(y) \);  
(ii) \( x = y \);  
(iii) \( \mathcal{L}(x) = \mathcal{L}(y) \).

Proof. Implications (ii) \( \Rightarrow \) (iii) and (iii) \( \Rightarrow \) (i) are trivial. To show (i) \( \Rightarrow \) (ii) assume \( x \neq y \), and choose, by 2.1, an \( A \in D(S) \) with \( x \in A' \) (\( = X \setminus A \), \( y \in A \). If \( f \in S \) with \( D(f) = A \), then \( f \in \mathcal{L}(x) \setminus \mathcal{L}(y) \), proving (i) \( \Rightarrow \) (ii).

Define a map \( \theta : X \to \{ \mathcal{L}(x) : x \in X \} \) via \( \theta(x) = \mathcal{L}(x) \), for each \( x \in X \). Clearly \( \theta \) is onto and 3.4 ensures \( \theta \) is 1-1. Hence the next lemma.

Lemma 3.5. \( \theta \) is a bijection.

Let \( \mathcal{P}_2 \) be the set of all doubletons \( \{ a, b \} \) in \( X \), \( a \neq b \).
DEFINITION 3.6. Given \( A \in \mathcal{P}_2 \), \( A = \{a, b\} \), let
\[
L(A) = \{l \in S : l(a) = l(b)\},
\]
\[
\mathcal{L}(A) = L(A) \cup (L(a) \cap L(b)).
\]
Then \( \mathcal{L}(A) \) is a left ideal of \( S \) which we call a set left ideal.

REMARK. It is convenient to extend Definitions 3.3 and 3.6 by letting
\[
\mathcal{L}(\Phi) = S.
\]

Recall that \( \pi(S) \) is normal for \( \mathcal{J}_X \)-normal \( S \) (Lemma 2.1). Thus \( L(A) = \Phi \) for some \( A \in \mathcal{P}_2 \) if and only if \( L(A) = \Phi \) for all \( A \in \mathcal{P}_2 \), i.e. if and only if \( S \subseteq \mathcal{J}_X \). If \( S \subseteq \mathcal{J}_X \) then \( \mathcal{L}(A) = L(a) \cap L(b) \) \((a, b \in A)\) is a degenerate set left ideal. The next lemma reveals that for any \( A = \{a, b\} \in \mathcal{P}_2 \), \( \mathcal{L}(a) \cap \mathcal{L}(b) \neq \Phi \), ensuring that \( \mathcal{L}(A) \neq \Phi \).

**LEMMA 3.7.** There exists an \( A \) in \( D(S) \) with \(|A'| \geq 2\).

**Proof.** Choose a proper partial transformation \( f \) in \( S \) and let \( x \in X \setminus D(f) \), \( y \in D(f) \), \( f(y) = z \). Take \( g \) in \( S \) with \( z \in X \setminus D(g) \) (by 2.1) and let \( t = gf \). Then \( x, y \in X \setminus D(t) \) and we let \( A = D(t) \).

REMARK 3.8. By applying the arguments of the proof of Lemma 3.7 to the map \( t \) instead of \( f \) it is easy to produce an \( A \in D(S) \) with \(|A'| \geq 3\).

**LEMMA 3.9.** Given \( A \) and \( B \) in \( \mathcal{P}_2 \), the following three statements are equivalent:

(i) \( \mathcal{L}(A) \subseteq \mathcal{L}(B) \);  
(ii) \( A = B \);  
(iii) \( \mathcal{L}(A) = \mathcal{L}(B) \).

**Proof.** Implications (ii) \( \Rightarrow \) (iii) and (iii) \( \Rightarrow \) (i) are trivial. We show (i) \( \Rightarrow \) (ii). Assume \( x \in B \setminus A \) and let \( C = (A \cup B) \setminus \{x\} \). Clearly, \(|C| \leq 3\). Using Remark 3.8 and the normality of \( D(S) \) (see 2.1) choose an \( f \) in \( S \) with \( x \in D(f) \) and \( C \subseteq X \setminus D(f) \). Then \( f \in \mathcal{L}(A) \setminus \mathcal{L}(B) \), so \( \mathcal{L}(A) \nsubseteq \mathcal{L}(B) \), proving (i) \( \Rightarrow \) (ii).

**NOTATION 3.10.** Given \( f \) and \( g \) in \( S \), let
\[
\Delta(f, g) = f(D(f) \setminus D(g)) \cup g(D(g) \setminus D(f)),
\]
\[
\bowtie(f, g) = \{ (f(x), g(x)) : x \in D(f) \cap D(g), f(x) \neq g(x) \}.
\]

**PROPOSITION 3.11.** Let \( f, g \in S \) with \( f \neq g \) and \( \mathcal{L}(f, g) \neq \{\Phi\} \). Then
\[
\mathcal{L}(f, g) = \left( \bigcap_{x \in \Delta(f, g)} \mathcal{L}(x) \right) \cap \left( \bigcap_{A \in \bowtie(f, g)} \mathcal{L}(A) \right).
\]

**Proof.** Let \( l \in \mathcal{L}(f, g) \), \( x \in \Delta(f, g) \) and without loss of generality let \( f(y) = x \) for some \( y \in D(f) \setminus D(g) \) (Notation 3.10). If \( x \in D(l) \), then \( lf = lg \) implies that \( lf(y) = lg(y) \), and so \( y \in D(g) \), a contradiction. Thus \( x \notin D(l) \) and
\[
l \in \mathcal{L}(x). 
\]
Now let \( A \in \mathcal{D}(f, g) \), \( A = \{ f(z), g(z) \} \). Then either \( l \in \mathcal{L}(f(z)) \cap \mathcal{L}(g(z)) \), or \( A \cap D(l) \neq \emptyset \), and \( lf = lg \) implies \( lf(z) = lg(z) \), whence \( l \in L(A) \). We conclude that

\[
l \in L(A). \tag{2}
\]

Since (1) and (2) hold for all \( x \in \Delta(f, g) \) and \( A \in \mathcal{D}(f, g) \), we deduce that

\[
\mathcal{L}(f, g) \subseteq \left( \bigcap_{x \in \Delta(f, g)} \mathcal{L}(x) \right) \cap \left( \bigcap_{A \in \mathcal{D}(f, g)} \mathcal{L}(A) \right).
\]

Conversely, let

\[
l \in \left( \bigcap_{x \in \Delta(f, g)} \mathcal{L}(x) \right) \cap \left( \bigcap_{A \in \mathcal{D}(f, g)} \mathcal{L}(A) \right).
\]

Firstly observe that

\[
D(lf) = D(lg). \tag{3}
\]

Indeed, assume that \( z \in D(lf) \setminus D(lg) \). Then \( z \in D(g) \) (otherwise \( f(z) \in \Delta(f, g) \)) and so \( l \in \mathcal{L}(f(z)) \), implying \( z \notin D(lf) \). Now \( f(z) \neq g(z) \) means that \( \{ f(z), g(z) \} = A \in \mathcal{D}(f, g) \), and so \( l \in \mathcal{L}(A) \). Since \( g(z) \notin D(l) \), we must also have that \( f(z) \notin D(l) \), or \( z \notin D(lf) \), a contradiction which proves (3).

Now take \( z \in D(lf) = D(lg) \). If \( f(z) = g(z) \), then certainly \( lf(z) = lg(z) \). If \( f(z) \neq g(z) \), then \( \{ f(z), g(z) \} = A \in \mathcal{D}(f, g) \). Since \( l \in \mathcal{L}(A) \) and \( A \subseteq D(l) \) we conclude that \( l \in L(A) \), or \( l \notin \mathcal{L}(f, g) \).

**Proposition 3.12.** Given an \( A \) in \( \mathcal{P}_2 \) and an \( x \) in \( X \) there exist \( f, g, p \) and \( q \) in \( S \) such that

\[
\mathcal{L}(A) = \mathcal{L}(f, g), \quad \mathcal{L}(x) = \mathcal{L}(p, q)
\]

and there is a \( k \) in \( S \) such that \( p = kf \), \( q = kg \).

*Proof.* Take an \( A \) in \( \mathcal{P}_2 \). On account of Proposition 3.11 it is sufficient to construct \( f \) and \( g \) such that \( D(f) = D(g) \) (and hence \( \Delta(f, g) = \emptyset \)) and \( \mathcal{D}(f, g) = \{ A \} \). Choose \( t \in S \) with \( A \subseteq X \setminus D(t) \) (by 3.7) and let \( c, d \in R(t) \), where \( c \neq d \) (note that \( S \) is constant-free). Let \( A = \{ a, b \} \) and \( s \in S \) take \( c \) to \( a \) and \( d \) to \( b \) (see 2.5). Then \( f = st \) and \( g = (a, b)f(a, b) = (a, b)f \) are the required transformations with \( \mathcal{L}(f, g) = \mathcal{L}(A) \).

Now let \( x \in X \) and choose \( k \in S \) such that \( k(a) = x \) and \( b \in X \setminus D(k) \). (To construct such \( k \) choose by 2.1 a map \( q \) in \( S \) with \( a \in D(q) \) and \( b \in X \setminus D(q) \), by 2.2 a map \( p \) in \( S \) which takes \( q(a) \) to \( x \), and let \( k = pq \).) It is easy to check that \( \mathcal{D}(kf, kg) = \emptyset \) and \( \Delta(kf, kg) = \{ x \} \), whence 3.11 ensures that \( \mathcal{L}(kf, kg) = \mathcal{L}(x) \). We let \( p = kf \), \( q = kg \).

We will show (Proposition 3.14) that each maximal function left ideal of \( S \) is either a point left ideal or a non-degenerate set left ideal, and these exhaust all maximal function left ideals.
LEMMA 3.13. For all \( A \) in \( \mathcal{P}_2 \) and \( x \) in \( X \):

(i) \( \mathcal{L}(x) \notin \mathcal{L}(A) \),

(ii) \( \mathcal{L}(A) \subseteq \mathcal{L}(x) \) implies \( \mathcal{L}(A) \) is degenerate.

Proof. (i) Let \( A = \{ a, b \} \) and assume that \( a \neq x \). With the aid of Lemmas 2.1 and 3.7 choose a \( B \in D(S) \) with \( a \in B \) and \( b, x \in B' \), together with \( f \in S \) such that \( D(f) = B \). Then \( f \in \mathcal{L}(x) \setminus \mathcal{L}(A) \).

(ii) If \( \mathcal{L}(A) = L(A) \cup (\mathcal{L}(a) \cap \mathcal{L}(b)) \subseteq \mathcal{L}(x) \), then \( L(A) \subseteq \mathcal{L}(x) \). Assume \( \mathcal{L}(A) \neq \Phi \), then \( x \notin A \) and each \( g \) such that \( A \cup \{ x \} \subseteq D(g) \) and \( g(a) = g(b) \) (chosen by Lemma 2.1) is in \( L(A) \setminus \mathcal{L}(x) \). Thus \( L(A) = \Phi \), and so \( \mathcal{L}(A) \) is degenerate.

PROPOSITION 3.14. Let \( f, g \in S \). Then \( \mathcal{L}(f, g) \) is a maximal function left ideal if and only if either \( \mathcal{L}(f, g) = \mathcal{L}(x) \), \( x \in X \), or \( \mathcal{L}(f, g) = \mathcal{L}(A) \), where \( \mathcal{L}(A) \) is non-degenerate, \( A \in \mathcal{P}_2 \).

Proof. Firstly, assume that \( \mathcal{L}(f, g) \) is a maximal function left ideal. Let \( x \in \Delta(f, g) \).

By 3.12 there exist \( p, q \in S \) such that \( \mathcal{L}(p, q) = \mathcal{L}(x) \). Hence \( \mathcal{L}(f, g) \subseteq \mathcal{L}(x) = \mathcal{L}(p, q) \) (by 3.11). The maximality of \( \mathcal{L}(f, g) \) implies

\[
\mathcal{L}(f, g) = \mathcal{L}(x) = \mathcal{L}(p, q).
\]

Similarly, if \( A \in D(f, g) \) then there are also \( t, s \in S \) with \( \mathcal{L}(t, s) = \mathcal{L}(A) \) (by 3.12) and \( \mathcal{L}(f, g) \subseteq \mathcal{L}(A) = \mathcal{L}(t, s) \) (by 3.11), implying that

\[
\mathcal{L}(f, g) = \mathcal{L}(A) = \mathcal{L}(t, s),
\]

because of the maximality of \( \mathcal{L}(f, g) \). Suppose \( \mathcal{L}(A) \) is degenerate, then for \( a \in A \), by 3.4,

\[
\mathcal{L}(f, g) = \mathcal{L}(A) \subseteq \mathcal{L}(a) = \mathcal{L}(l, r),
\]

for some \( l, r \in S \) (by 3.12), a contradiction to the maximality of \( \mathcal{L}(f, g) \).

For the converse, assume that \( \mathcal{L}(f, g) = \mathcal{L}(x) \), for some \( x \in X \). To show that \( \mathcal{L}(f, g) \) is maximal suppose that there are \( p, q \in S \) with \( \mathcal{L}(p, q) \supseteq \mathcal{L}(f, g) \), that is, by 3.11,

\[
\mathcal{L}(x) = \mathcal{L}(f, g) \subseteq \mathcal{L}(p, q) = \left( \bigcap_{y \in \Delta(p, q)} \mathcal{L}(y) \right) \cap \left( \bigcap_{B \in D(p, q)} \mathcal{L}(B) \right). \tag{4}
\]

If \( D(p, q) \neq \Phi \), then \( \mathcal{L}(x) \subseteq \mathcal{L}(B) \), for every \( B \in D(p, q) \), contradicting 3.13(i). Thus \( D(p, q) \) is empty and, for every \( y \in \Delta(p, q) \), \( \mathcal{L}(x) \subseteq \mathcal{L}(y) \). Lemma 3.4 ensures that \( \Delta(p, q) = \{ x \} \) and we deduce from (4) that \( \mathcal{L}(f, g) = \mathcal{L}(p, q) \).

Finally assume that \( \mathcal{L}(f, g) = \mathcal{L}(A) \), \( A \in \mathcal{P}_2 \), and \( \mathcal{L}(A) \) is non-degenerate. If \( \mathcal{L}(f, g) \subseteq \mathcal{L}(t, s) \) for \( t, s \in S \), then 3.11 implies

\[
\mathcal{L}(A) = \mathcal{L}(f, g) \subseteq \mathcal{L}(t, s) = \left( \bigcap_{z \in \Delta(t, s)} \mathcal{L}(z) \right) \cap \left( \bigcap_{C \in D(p, q)} \mathcal{L}(C) \right). \tag{5}
\]

If \( \Delta(t, s) \neq \Phi \), then \( \mathcal{L}(A) \subseteq \mathcal{L}(z) \), for each \( z \in \Delta(t, s) \), contradicting 3.13(ii). Hence
\[ \Delta(t, s) = \Phi \text{ and, for each } C \in \mathcal{D}(p, q), \mathcal{L}(A) \subseteq \mathcal{L}(C). \] Thus \( \mathcal{D}(p, q) = \{ A \} \) (3.9) and we deduce from (5) that \( \mathcal{L}(f, g) = \mathcal{L}(t, s) \).

It is clear from 3.2 that each automorphism \( \phi \) of \( S \) permutes maximal function left ideals. Our aim is to show that \( \phi \) also permutes point left ideals. If all the set left ideals are degenerate, that is \( S \subseteq \mathcal{I}_x \), then, as the above proposition reveals, the point left ideals are the only maximal function left ideals. In the next proposition we formulate a property which distinguishes the non-degenerate set left ideals and is preserved under \( \phi \).

**Proposition 3.15.** Let \( S \notin \mathcal{I}_x \) and \( \mathcal{L}(f, g) \) be a maximal function left ideal. Then \( \mathcal{L}(f, g) \) is a set left ideal if and only if

\[ \forall \text{ maximal function left ideal } L \exists k \in S \text{ such that } \mathcal{L}(kf, kg) = L. \quad (6) \]

**Proof.** Assume firstly that \( \mathcal{L}(f, g) = \mathcal{L}(A) \) (non-degenerate), \( A = \{ a, b \} \in \mathcal{P}_2 \). We show that (6) holds. If \( L = \mathcal{L}(x) \), for some \( x \in X \), then we appeal to Lemma 3.12. Hence assume \( L = \mathcal{L}(B) \), for some \( B \in \mathcal{P}_2 \). Choose \( k \in S \) mapping \( A \) onto \( B \) (by 2.5). Then \( D(kf) = D(kg) \) and so \( \Delta(kf, kg) = \Phi \). (Indeed, assume, for example, that \( u \in D(kf) \setminus D(kg) \). Then \( u \in D(f) = D(g) \), since \( \Delta(f, g) = \Phi \), by 3.11 and 3.13(ii), \( f(u) \in D(k) \) and \( g(u) \notin D(k) \). Thus \( f(u) \neq g(u) \), so that by Lemma 3.9 \( \{ f(u), g(u) \} = A \) and \( \Delta(f, g) = \Phi \), a contradiction.) Also, \( D(kf, kg) = \{ B \} \), since \( kf(u) \neq kg(u) \), for some \( u \in D(kf) \), implies that \( f(u) \neq g(u) \), or \( \{ f(u), g(u) \} = A \), again by 3.9, and so by the choice of \( k \), \( \{ kf(u), kg(u) \} = B \). Proposition 3.11 ensures that \( \mathcal{L}(kf, kg) = \mathcal{L}(B) \), proving (6).

For the converse, assume that \( \mathcal{L}(f, g) \) satisfies (6) and is a point left ideal \( \mathcal{L}(x) \) (Proposition 3.14). Let \( L = \mathcal{L}(A) \), \( A = \{ a, b \} \in \mathcal{P}_2 \), be a non-degenerate set left ideal (recall, \( S \notin \mathcal{I}_x \)), and \( k \in S \) be such that \( \mathcal{L}(kf, kg) = \mathcal{L}(A) \). Then by 3.11 and 3.13(ii), \( \Delta(kf, kg) = \Phi \), that is \( D(kf) = D(kg) \). Since \( \mathcal{L}(fg) = \mathcal{L}(x) \), it follows from 3.11 and 3.13(i) that \( \Delta(f, g) = \Phi \). Assume without loss of generality that \( x = f(y) \), where \( y \in D(f) \setminus D(g) \). If \( x \in D(k) \), then \( y \in D(kf) = D(kg) \subseteq D(g) \), a contradiction. Hence \( x \notin D(k) \) and so \( k \in \mathcal{L}(x) \), which means that \( kf = kg \), a contradiction to the assumption that \( \mathcal{L}(kf, kg) = \mathcal{L}(A) \).

**Proposition 3.16.** Let \( \phi \in \text{Aut } S \). Given \( x \in X \) there exists \( y \in X \) such that \( \phi(\mathcal{L}(x)) = \mathcal{L}(y) \).

**Proof.** Let \( x \in X \) and choose \( f, g \in S \) with \( \mathcal{L}(f, g) = \mathcal{L}(x) \) (by 3.12). Proposition 3.14 ensures that \( \mathcal{L}(f, g) \) is a maximal function left ideal. Whence

\[ \phi(\mathcal{L}(x)) = \phi(\mathcal{L}(f, g)) = \mathcal{L}(\phi(f), \phi(g)) \quad (by \ 3.2) \]

is a maximal function left ideal. If \( S \) contains only degenerate set left ideals then \( \mathcal{L}(\phi(f), \phi(g)) = \mathcal{L}(y) \) as required. Hence assume that there are non-degenerate set left ideals. Since \( \mathcal{L}(f, g) = \mathcal{L}(x) \), by 3.15 there exists a maximal function left ideal \( L \) such that for any \( k \in S \), \( \mathcal{L}(kf, kg) \neq L \), or for any \( k' \in S \), \( \mathcal{L}(k' \phi(f), k' \phi(g)) \neq \phi(L) \). With the aid of 3.2 we deduce that \( \phi(L) \) is a maximal function left ideal. Then 3.15 ensures that \( \mathcal{L}(\phi(f), \phi(g)) = \mathcal{L}(y) \), for some \( y \in X \).
Using the above proposition define a map

$$\eta: \{L(x): x \in X\} \to \{L(x): x \in X\} \quad \text{via} \quad \eta(L(x)) = \phi(L(x)), $$

for each $L(x)$. Similarly, by considering the automorphism $\phi^{-1}$, define a map

$$\xi: \{L(x): x \in X\} \to \{L(x): x \in X\} \quad \text{via} \quad \xi(L(x)) = \phi^{-1}(L(x)).$$

Certainly $\xi$ is the inverse of $\eta$ and so we have proved the following.

**Lemma 3.17.** $\eta$ is a bijection.

By Lemma 3.4, $L(x) = L(y)$ if and only if $x = y$ ($x, y \in X$). We can therefore now define a map $h: X \to X$ by $h(x) = y$, where $y$ is given by $\eta(L(x)) = L(y)$, for $x \in X$. Thus, with the notation of 3.5,

$$h = \theta^{-1}\eta\theta.$$  

By 3.17, $h$ is a bijection; that is, $h \in G_X$. We call $h$ the bijection associated with $\phi$.

Now we will prove the main result of this paper.

**Theorem 3.18.** If $S$ is a $G_x$-normal subsemigroup of $P_X$, then $\text{Aut } S = \text{Inn } S$.

**Proof.** If $S$ consists of total transformations we appeal to [3, Theorem 1.1]. If $S$ contains a constant map, the result is given in [11, Theorem 2]. Thus we assume that $S$ is a constant-free semigroup containing a proper partial transformation, and so $L(x) \neq \Phi$ for every $x \in X$.

Take $f \in S$, $x \in D(f)$ and let $f(x) = y$. Since $f \notin L(x)$, also $\phi(f) \notin \eta(L(x)) = L(h(x))$, where $h$ is the bijection associated with $\phi$. Hence $h(x) \in D(\phi(f))$.

Now observe that for any $k$ in $L(y)$, $kf \in L(x)$, hence for any $k'$ in $L(h(y))$, $k' \phi(f) \in L(h(x))$. Let $\phi(f)h(x) = z$. If $z \neq h(y)$, we can always choose $k'$ in $L(h(y))$ with $z \in D(k')$ (Lemma 2.1). But then $k'\phi(f) \notin L(h(x))$, a contradiction which shows that $z = h(y)$. Thus

$$\phi(f)h(x) = h(y) = hf(x).$$

Since this is true for all $x$ in $D(f)$, we conclude that

$$\phi(f) = hfh^{-1},$$

and, since $f$ is an arbitrary element of $S$, the result follows.

**References**

