The problem of describing all automorphisms of a given semigroup of transformations of a set \( X \) has interested a number of mathematicians in the past fifty years. In 1937 Schreier [10] showed that every automorphism of the full transformation semigroup \( T_X \) is inner (that is, acts as a conjugation by some bijection of \( X \)). In 1952 Mal'cev [7] generalized this result by showing that every ideal of \( T_X \) has only inner automorphisms. More recently Symons [11] showed that all automorphisms of any \( G_X \)-normal semigroup (that is, invariant under a conjugation by any bijection of \( X \)) over a finite set \( X \) are inner, while Schein [9] produced the same result for \( G_X \)-normal semigroups of one-to-one transformations over an infinite set \( X \). (See [2] for the special base of Baer-Levi semigroups.)

Chapters 2 and 3 of this thesis constitute a contribution towards the solution of the problem of describing all automorphisms of a given semigroup of transformations of an infinite set \( X \). In Chapter 2 (see also [4]) we extend the well-known result from group theory, namely that any normal group of bijections of an infinite set \( X \) has only inner automorphisms, to an analogous one in semigroup theory. We show that any \( G_X \)-normal semigroup of transformations of an infinite set \( X \) has only inner automorphisms. Our purpose in Chapter 3 (see also [3]) is to offer a complete description of all automorphisms of an arbitrary Croisot-Teissier
semigroup [1], a task suggested by Schein. In this joint work with O'Meara and Wood a rich variety of automorphisms is found, ranging from inner, to "locally" inner, to thoroughly outer. We also present a description of Green's relations on Croisot-Teissier semigroups.

In Chapter 4 (and in [5]) we are concerned with the problem of a characterisation of all subsets of the power set of $X$, $P_X$, which serve as sets of ranges of semigroups of transformations of $X$. This problem was suggested by Schein and to our knowledge has been solved only for the case of monogenic semigroups of partial transformations by Olonichev [8]. We define a normal subset of $P_X$ and characterise all normal subsets of $P_X$ which serve as sets of ranges of semigroups of total transformations of $X$. In particular, we give necessary and sufficient conditions for a subset of $P_X$ to be the set of ranges of a $G_X$-normal and a constant-free $G_X$-normal semigroup of total transformations.

In Chapter 5 (and in [6]) for a particular normal subset of $P_X$ we give necessary and sufficient conditions for an order-automorphism to be determined by a bijection of $X$ (that is, induced). We then characterise those normal subsets of $P_X$ for which all order-automorphisms are induced. Apart from being of independent interest, this problem is connected with the study of automorphisms of transformation semigroups. For if an automorphism $\phi$ of a transformation semigroup $S$ is inner, then $\phi$ produces an induced order-automorphism of the set $R(S)$ of ranges of all transformations in $S$. On the other hand, in instances where an automorphism $\phi$ of $S$ yields an order-automorphism of $R(S)$, the knowledge that all order-automorphisms of $R(S)$ are induced can be a first step in showing that $\phi$ is inner.

References

Automorphisms and range families


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