Solution to the problem 11366 of the American Mathematical Monthly, May Issue 2008

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Problem 11357 Proposed by Nicolae Naghel, University of North Texas, Denton, TX. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous differentiable function such that $\phi(0) = 0$ and ϕ' is strictly increasing. For $a > 0$, let C_a denote the space of all continuous functions from $[0, a]$ into $\mathbb{R}$, and for $f \in C_a$, let $I(f) = \int_0^a (\phi(x)f(x) - x\phi(f(x)))dx$. Show that I has a finite supremum on C_a and that there exists an $f \in C_a$ at which supremum is attained.*

Solution: For every $x \in [0, a]$ we let $g_x(u) = \phi(x)u - x\phi(u)$ defined for all $u \in \mathbb{R}$. Consider the derivative of this function: $g'_x(u) = \phi(x) - x\phi'(u)$. By the Mean Value Theorem we know that $\phi(x) = \phi(x) - \phi(0) = (x - 0)\phi'(c_x)$ for some c_x between 0 and x. If $x > 0$, then $g'_x(u) = x(\phi'(c_x) - \phi'(u))$. By our assumption on ϕ' we see that g_x has a maximum attained at $u = c_x$. Also, because ϕ' is strictly increasing, c_x is unique when $x > 0$. If $x = 0$ we see that $g_x \equiv 0$ and we simply define $c_0 = 0$. This gives a function $x \to c_x$ which we denote by f_0. Clearly

$$f_0(x) = \begin{cases} \phi'^{-1}(\frac{\phi(x)}{x}) \quad if \quad 0 < x \le a \\ \\ 0 \quad if \ x = 0. \end{cases}$$

This function is continuous at every point $x > 0$, since ϕ' is continuous and strictly increasing. Also, because $0 < c_x < x$ for $x > 0$, this function is continuous at 0 too. So, $f \in C_a$.

Therefore for every $f \in C_a$ we have

$$I(f) = \int_0^a g_x(f(x))dx \leq \int_0^a g_x(f_0(x))dx = I(f_0).$$

This inequality answers both questions in the statement of the problem. ■

Comments: For a simple function like $\phi(x) = x^n$ with $n = 2m$, $m \in \mathbb{N}$, one can find $f_0(x) = \frac{x}{n^{\frac{1}{n-1}}}$ and the constant $I(f_0)$. So the problem shows that

$$\int_0^a [x^n f(x) - xf(x)^n]dx \leq \frac{(n-1)a^{n+1}}{(n+1)n^{\frac{n}{n-1}}},\ f \in C_a.$$

It is an interesting problem to calculate this constant for other naturally defined functions ϕ. For example, if $\phi(x) = e^x - 1$ we get

$$f_0(x) = \begin{cases} \ln(\frac{e^x-1}{x}) & if \quad 0 < x \leq a \\ \\ 0 \quad if \ x = 0. \end{cases}$$

For $a = 1$ we get $I(f_0) = \dfrac{5}{2} - e + \displaystyle\int_0^1 (e^x - 1)\ln\frac{e^x-1}{x}dx \approx 0.4766693501.$ So, the inequality we get can be written as

$$\int_0^1 (e^x - 1)\left(f(x) - \ln\frac{e^x-1}{x}\right) - xe^{f(x)}dx \leq 2 - e,\ f \in C_1.$$