Solution to Problem 955 of The College Mathematics Journal, vol. 42, No. 3, May 2011

Eugen J. Ionascu
Department of Mathematics, Columbus State University,
4225 University Dr., Columbus, GA 31907

August 4th, 2011

Problem 898. *Proposed by José Luis Díaz-Barrero, Universitat Politécnica de Catalunya, Barcelona, Spain.* Let a, b, $c > 0$ and $a + b + c = 3$. Prove that

$$\sum_{cycl} \frac{bc}{\sqrt[4]{a^2+3}} \le \frac{3\sqrt{2}}{2} \tag{1}$$

where the sum is over all cyclic permutation of (a, b, c) and equality occurs when $a = b = c = 1$.

Solution: First let us observe that $x^2 + 3 \ge \frac{(x+3)^2}{4}$, which is equivalent to $(x-1)^2 \ge 0$. Using this inequality for the denominators in (1), we observe that (1) follows from the following inequality:

$$\sum_{cycl} \frac{bc}{\sqrt{a+3}} \le \frac{3}{2}. \tag{2}$$

One more application of the same idea shows that, (2) follows from the inequality:

$$\sum_{cycl} \frac{2bc}{\sqrt{a}+3} \le \frac{3}{2}. \tag{3}$$

So, we will concentrate in proving (3) which is a stronger inequality than the one required in the problem.

Multiplying (3) by 6 and substituting $9 = (a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ac$, (3) changes into

$$a^2+b^2+c^2-2ab-2bc-2ac+\sum_{cycl} 4bc\left(1-\frac{3}{\sqrt{a}+3}\right) \geq 0 \Leftrightarrow$$

$$a^2+b^2+c^2-2ab-2bc-2ac+4abc\sum_{cycl}\frac{1}{\sqrt{a}(\sqrt{a}+3)} \geq 0 \Leftrightarrow$$

$$a^2+b^2+c^2-2ab-2bc-2ac+3abc+4abc\left(\sum_{cycl}\frac{1}{\sqrt{a}(\sqrt{a}+3)}-\frac{3}{4}\right) \geq 0. \quad (4)$$

In order to prove (4), we will show the following two inequalities:

$$a^2+b^2+c^2-2ab-2bc-2ac+3abc \geq 0, \ \textit{and} \quad (5)$$

$$\sum_{cycl}\frac{1}{\sqrt{a}(\sqrt{a}+3)} \geq \frac{3}{4}. \quad (6)$$

To prove (6), we consider the function $g(x) = \frac{1}{\sqrt{x}(\sqrt{x}+3)}$ defined for $x > 0$. One can check that the second derivative of g is given by

$$g''(x) = \frac{1}{4}(8x+27\sqrt{x}+27)(\sqrt{x}+3)^{-3}x^{-5/2} > 0, \ x > 0.$$

This shows that g is a convex function and so $\frac{1}{3}(g(a)+g(b)+g(c)) \geq g(\frac{a+b+c}{3}) = g(1) = \frac{1}{4}$. This last inequality is equivalent with (6).

Now, to show (5), let us observe that this inequality is equivalent to

$$(a+b+c)(a^2+b^2+c^2-2ab-2bc-2ac)+9abc \geq 0 \Leftrightarrow$$

$$a^3+b^3+c^3-a^2(b+c)-b^2(a+c)-c^2(a+b)+3abc \geq 0. \quad (7)$$

Without loss of generality let us assume that $a \leq b \leq c$. Since $3 = a+b+c \geq 3a$, we see that $a \leq 1$. Now, (7) can be written as

$$a(b-a)(c-a)+(b-c)^2(b+c-a) \geq 0 \ \textit{or}$$

$$a(b-a)(c-a)+(b-c)^2(3-2a) \geq 0,$$

which is true under our assumption. ■