Solution to problem 947, CMJ, Vol.42, No. 2, March 2011, Page 151

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Problem 947: Find all non-constant polynomials P and Q such that

$$\prod_{i=1}^{n} P(i) = Q\left(\prod_{i=1}^{n} i\right),$$

for all integers $n \geq 1$.

Solution:

The only polynomials with the desired properties are $P(x) = Q(x) = x^m$ for any positive integer m.

To see this, let k and m be positive integers and denote $P(x) = p_k x^k + ... + p_0$ with $p_k \neq 0$ and $Q(x) = q_m x^m + ... + q_0$ with $q_m \neq 0$. Now, since $P(n) = \dfrac{Q(n!)}{Q((n-1)!)}$ for $n \geq 2$,

$$\lim_{n\to\infty} \frac{Q(n!)}{P(n)Q((n-1)!)} = \lim_{n\to\infty} \frac{q_m (n!)^m}{p_k (n)^k \cdot q_m ((n-1)!)^m} = \lim_{n\to\infty} \frac{n^m}{p_k (n)^k} = 1.$$

So, $p_k = 1$ and $m = k$. Now, since $P(x) = x^m + (p_{m-1})x^{m-1} ... + p_0$,

$$\lim_{n\to\infty} [(p_{m-1})n^{m-1} + ... + p_0] = \lim_{n\to\infty} [P(n) - n^m] = \lim_{n\to\infty} \left[\frac{Q(n!)}{Q((n-1)!)} - n^m\right] =$$

$$\lim_{n\to\infty} \left[\frac{Q(n!) - Q((n-1)!)n^m}{Q((n-1)!)}\right] = \lim_{n\to\infty} \frac{(q_{m-1})(1-n)(n!)^{m-1} + ... + (1-n^m)q_0}{(q_m)((n-1)!)^m + ... + q_0} = 0.$$

Thus, $p_0 = ... = p_{m-1} = 0$, which yields $P(x) = x^m$. Now, $P(1)P(2)\cdots P(n) = (n!)^m = Q(n!)$, so $Q(x) = x^m$ for infinitely many values of x. This is enough to conclude that $Q(x) - x^m$ must be the zero polynomial. Therefore, $P(x) = Q(x) = x^m$.