Solution to Problem 878 of The College Mathematics Journal, May Issue 2008

Eugen J. Ionascu

Department of Mathematics, Columbus State University,

4225 University Dr., Columbus, GA 31907

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Problem 878. *Proposed by Jose Luis Diaz-Barrero, Universidad Politecnica de Cataluna, Barcelona, Spain* Let a, b, and c be the lengths of the sides and s the semi-perimeter of the triangle ABC. Prove that

$$(a+b-c)^{a+b+s}(a-b+c)^{b+c+s}(-a+b+c)^{c+a+s} \le a^{\frac{a}{2}+2s}b^{\frac{b}{2}+2s}c^{\frac{c}{2}+2s}. \quad (1)$$

Solution: In [1] the author of this problem describes a way of deriving inequalities of this type. In this case we look at the function $f(x) = (2s + \frac{x}{2}) \ln x$ and check that

$$f''(x) = \frac{x-4s}{2x^2} < 0, for\ all\ x < 4s.$$

This is a concave function and so one can use Karamata's Inequality for the triples $(a+b-c, a-b+c, -a+b+c)$ and (a, b, c). Indeed, without loss of generality we may assume $a \ge b \ge c$ and see that the first triple majorizes the second:

$$a+b-c \ge a, \quad a+b-c+a-b+c \ge a+b,$$

$$(a+b-c)+(a-b+c)+(-a+b+c) = a+b+c.$$

Karamata's Inequality or the Majorization Inequality (see [2] and [3]) states that if $(x_1, ..., x_n)$ majorizes $(y_1, y_2, ..., y_n)$ and f is a concave function whose domain contains the points x_i and y_i then

$$\sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} f(y_i).$$

Clearly, the above values are in the concavity interval of f. Hence

$$f(a+b-c)+f(a-b+c)+f(-a+b+c) \leq f(a)+f(b)+f(c),$$

which is easily seen to be equivalent to the required inequality (1). ■

References

[1] J. Luis Diaz-Barrero, *Some cyclical inequalities for the triangle*, J. Ineq. Pure and Appl. Math. 6(1) Art 20, 2005

[2] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, Academic Press, NY, 1979

[3] C. Niculescu and F. Popovici, *The extension of majorization inequalities within the framework of relative convexity*, J. Ineq. Pure and Appl. Math. 7(1) Art 27, 2006