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On the Unitary Systems Affiliated with Orthonormal Wavelet Theory in *n*-Dimensions

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We consider systems of unitary operators on the complex Hilbert space $L^2(\mathbb{R}^n)$ of the form $\mathcal{U} := \mathcal{U}_{D_A, T_{v_1}, ..., T_{v_n}} := \{D^m T_{v_1}^{\ell_1} \cdots T_{v_n}^{\ell_n} : m, \ell_1, ..., \ell_n \in \mathbb{Z}\}$, where D_A is the unitary operator corresponding to dilation by an $n \times n$ real invertible matrix A and $T_{v_1}, ..., T_{v_n}$ are the unitary operators corresponding to translations by the vectors in a basis $\{v_1, ..., v_n\}$ for $\mathbb{R}^n$. Orthonormal wavelets ψ are vectors in $L^2(\mathbb{R}^n)$ which are complete wandering vectors for $\mathcal{U}$ in the sense that $\{U\psi : U \in \mathcal{U}\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. It has recently been established that whenever A has the property that all of its eigenvalues have absolute values strictly greater than one (the expansive case) then $\mathcal{U}$ has orthonormal wavelets. The purpose of this paper is to determine when two $(n+1)$-tuples of the form $(D_A, T_{v_1}, ..., T_{v_n})$ give rise to the "same wavelet theory." In other words, when is there a unitary transformation of the underlying Hilbert space that transforms one of these unitary systems onto the other? We show, in particular, that two systems $\mathcal{U}_{D_A, T_{e_i}}$, and $\mathcal{U}_{D_B, T_{e_i}}$, each corresponding to translation along the coordinate axes, are unitarily equivalent if and only if there is a matrix C with integer entries and determinant ± 1 such that $B = C^{-1}AC$. This means that different expansive dilation factors nearly always yield unitarily inequivalent wavelet theories. Along the way we establish necessary and sufficient conditions for an invertible real $n \times n$ matrix A to have the property that the dilation unitary operator D_A is a bilateral shift of infinite multiplicity. © 1998 Academic Press

1. INTRODUCTION

The mathematical concept of an orthonormal wavelet in $L^2(\mathbb{R})$ has become extremely useful in practical applications to signal processing involving filtering, detection, data compression, etc. In fact, the use of wavelet technology in signal processing is now a big business, and is growing rapidly.

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The existence of a single-function orthonormal wavelet in $L^2(\mathbb{R}^n)$ for $n>1$ (see below for definition) has only very recently been established [6]. In that paper the authors showed that there exists a measurable set $S\subset\mathbb{R}^n$ such that the inverse Fourier transform of the normalized characteristic function χ_S is an orthonormal single-function wavelet in $L^2(\mathbb{R}^n)$. Moreover the authors establish the existence of such wavelets for unitary systems associated with more general dilations than the usual dyadic one (see below for definitions). See also [13] for several concrete examples of single-function wavelets in the plane. Thus it would seem to be of interest to know when two unitary systems (see below for definition) acting on $L^2(\mathbb{R}^n)$ for some $n\geqslant 1$ give rise to the "same" wavelet theory. In this article we settle this question completely for $n=1$ (Proposition 2.1), and make substantial progress on it for $n>1$ (Theorems 3.1 and 5.8).

We begin by introducing some preliminary terminology and notation. Let $\mathbb{R}^n$ be, as usual, n-dimensional Euclidian space and denote by $L^2(\mathbb{R}^n)$ the complex Hilbert space of (equivalence classes of) square integrable complex-valued functions on $\mathbb{R}^n$ relative to Lebesgue-Borel measure μ_n on $\mathbb{R}^n$. We write $B(L^2(\mathbb{R}^n))$ for the algebra of all bounded linear operators on $L^2(\mathbb{R}^n)$ and we say, following [5], that a set $\mathscr{U}$ of unitary operators in $B(L^2(\mathbb{R}^n))$ is a *unitary system* if the identity operator I on $L^2(\mathbb{R}^n)$ belongs to $\mathscr{U}$. A *complete wandering vector* for a unitary system $\mathscr{U}$ is a unit vector $f\in L^2(\mathbb{R}^n)$ such that $\mathscr{U}f=\{Uf : U\in\mathscr{U}\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. Unitary systems pertinent to orthonormal wavelet theory in higher dimensions arise in the following way. For $k\in\mathbb{N}$, let $M_k(\mathbb{R})$ $[M_k(\mathbb{C})]$ denote the algebra of $k\times k$ matrices with entries from $\mathbb{R}$ $[\mathbb{C}]$, and let $M'_k(\mathbb{R})$ $[M'_k(\mathbb{C})]$ denote the group of invertible matrices in $M_k(\mathbb{R})$ $[M_k(\mathbb{C})]$. If $A, B\in M'_n(\mathbb{R})$ and $v, w\in\mathbb{R}^n$ then the operators D_A, T_v in $B(L^2(\mathbb{R}^n))$ defined by

$$(D_A f)(x)=|\det A|^{1/2} f(Ax), \qquad x\in\mathbb{R}^n, \quad f\in L^2(\mathbb{R}^n), \tag{1}$$

$$(T_v f)(x)=f(x-v), \qquad x\in\mathbb{R}^n, \quad f\in L^2(\mathbb{R}^n), \tag{2}$$

are clearly unitary and satisfy the following relations:

$$D_A D_B=D_{BA}, \qquad A, B\in M'_n(\mathbb{R}), \tag{3}$$

$$T_{nv+mw}=T_v^n T_w^m, \qquad m, n\in\mathbb{N}, \quad v, w\in\mathbb{R}^n, \tag{4}$$

$$T_v D_A=D_A T_{Av}, \qquad v\in\mathbb{R}^n, \quad A\in M'_n(\mathbb{R}). \tag{5}$$

Let $\{v_1, ..., v_n\}$ be a basis (not necessarily orthonormal) for $\mathbb{R}^n$, and consider the unitary system

$$\mathscr{U}_{D_A, T_{v_i}}:=\{D_A^m T_{v_1}^{\ell_1}\cdots T_{v_n}^{\ell_n} : m, \ell_1, ..., \ell_n\in\mathbb{Z}\}. \tag{6}$$

If $f \in L^2(\mathbb{R}^n)$ is a complete wandering vector for the above unitary system, then f is called a (single-function) *orthonormal wavelet*, and the collection of all such wavelets relative to this unitary system will be denoted by $\mathscr{W}_{A, v_i}$. It is known [6] that wavelets exist (although they may not have good smoothness properties [1]) if either all of the eigenvalues of the matrix A in (6) have modulus greater than 1 (i.e., A is *expansive*) or all have modulus less than 1 (i.e., A^{-1} is expansive). (It seems not to be known, however, whether the existence of wavelets for a unitary system of the form (6) implies the expansivity of A or A^{-1}.)

We note that wavelet theory also includes the notion of a *wavelet family* (cf. [8, 13]) which is a p-tuple $\{\psi_1, \psi_2, ..., \psi_p\}$, $p \in \mathbb{N}$, of functions in $L^2(\mathbb{R}^n)$ such that

$$\mathscr{U}_{D_A, T_{v_i}}\{\psi_i\} := \{D_A^m T_{v_1}^{\ell_1} \cdots T_{v_n}^{\ell_n} \psi_j : m, \ell_1, ..., \ell_n \in \mathbb{Z}, 1 \leqslant j \leqslant p\},$$

is an orthonormal basis for $L^2(\mathbb{R}^n)$. (The term *multi-wavelet* is also used in the literature for wavelet family. Moreover, the word *family* or the prefix *multi* is often dropped, and such a family is simply called a *wavelet*.) Equivalently, from a functional analytic point of view, $\{\psi_1, \psi_2, ..., \psi_p\}$ is a wavelet family if and only if $\bigvee \{\psi_1, \psi_2, ..., \psi_p\}$ is a complete wandering subspace for $\mathscr{U}_{D_A, T_{v_i}}$. Wavelet families are meaningful even for dimension $n = 1$, and are frequently considered in the literature. Until very recently orthonormal wavelet theory in $\mathbb{R}^n$ has been concerned *only* with wavelet families, and only very special dilation matrices $A \in M'_n(\mathbb{R})$ have been considered, most notably $A = 2I$ (the *dyadic* case; see, for instance, [9]). Perhaps this is because the existence of single-function orthonormal wavelets in $L^2(\mathbb{R}^n)$ for $n > 1$ was thought to be impossible before the publication of [6]. As noted above, in [6] it was shown that if A is *any* expansive matrix and $\{v_1, v_2, ..., v_n\}$ is an arbitrary basis for $\mathbb{R}^n$, then single-function wavelets for $\mathscr{U}_{D_A, T_{v_i}}$ exist. So all of the systems $\mathscr{U}_{D_A, T_{v_i}}$, with A expansive, are affiliated with wavelet theory in n-dimensions. In addition, it is entirely possible (we have no solid evidence either way) that single-function wavelets exist for some of the non-expansive dilation matrices. For these reasons, it seems desirable to classify these known unitary systems in order to answer the basic question of which systems give rise to the "same" wavelet theory. This question makes sense whether one is interested only in single-function wavelets or, more generally, in wavelet families. Our results to follow are valid in both these situations.

There is a natural equivalence relation on the set of all $(n+1)$-tuples of the form $(D_A, T_{v_1}, ..., T_{v_n})$ (where $A \in M'_n(\mathbb{R})$ and $\{v_1, ..., v_n\}$ is a basis of $\mathbb{R}^n$) that makes precise the question: when do two $(n+1)$-tuples give rise to the same wavelet theory? We will use the abbreviation (D_A, T_{v_i}) for $(D_A, T_{v_1}, ..., T_{v_n})$.

DEFINITION 1.1. Two $(n+1)$-tuples (D_A, T_{v_i}) and (D_B, T_{w_i}) are said to be (unitarily) equivalent (written "$\sim$") if there exists a unitary $U \in B(L^2(\mathbb{R}^n))$ such that

$$D_B = U^* D_A U \tag{7}$$

and

$$T_{w_i} = U^* T_{v_i} U, \qquad i = 1, ..., n. \tag{8}$$

It is obvious that if $(D_A, T_{v_i}) \sim (D_B, T_{w_i})$ then we have $U\mathcal{W}_{B, w_i} = \mathcal{W}_{A, v_i}$, where U is a unitary operator satisfying (7) and (8). Moreover, if S is a matrix in $M'_n(\mathbb{R})$, then $(D_{SAS^{-1}}, T_{Sv_i}) \sim (D_A, T_{v_i})$. One of the main results of this paper is to obtain (Theorem 3.1) the converse of this last assertion. Namely, we show that if $(D_A, T_{v_i}) \sim (D_B, T_{w_i})$, then $B = C^{-1}AC$, where C is the matrix of the linear transformation $\tilde{C}$ defined on the basis $\{w_1, ..., w_n\}$ by $\tilde{C}w_i = v_i$, written realative to the canonical basis for $\mathbb{R}^n$. (In particular, if $w_i = v_i$ for all i, but $A \neq B$, then (D_A, T_{v_i}) and (D_B, T_{w_i}) are not equivalent.) We also show that every $(n+1)$-tuple is equivalent to a unique one of the form $(D_A, T_{e_1}, ..., T_{e_n})$ where $\{e_1, ..., e_n\}$ is the canonical basis for $\mathbb{R}^n$.

Another possible relation between two $(n+1)$-tuples (D_A, T_{v_i}) and (D_B, T_{w_i}) is given by the following definition.

DEFINITION 1.2. Two $(n+1)$-tuples (D_A, T_{v_i}) and (D_B, T_{w_i}) are said to be weakly equivalent (written "$\approx$") if there exists a unitary $V \in B(L^2(\mathbb{R}^n))$ such that

$$V\mathcal{U}_{A, v_i} V^* = \mathcal{U}_{B, w_i}$$

(meaning only, of course, that the mapping $Z \to VZV^*$ takes the set $\mathcal{U}_{A, v_i}$ onto the set $\mathcal{U}_{B, w_i}$ without regard to the images of D_A and the T_{v_i}).

Our second main result (Theorem 5.8) characterizes the relation of weak equivalence, at least for certain pairs of matrices A and B in $M'_n(\mathbb{R})$.

It also turns out (see Section 4) that if a unitary system $\mathcal{U}_{D_A, T_{v_i}}$ admits a wavelet, then the unitary operator D_A is a bilateral shift of infinite multiplicity. We characterize (Theorem 4.2) exactly those $A \in M'_n(\mathbb{R})$ for which D_A is such a bilateral shift.

2. THE CASE $n = 1$

The one-dimensional case is significantly simpler than the general case. For that reason we consider it separately. In this case the matrix A and the vector v in (1) and (2) are nonzero real numbers a and b, and (1), (2) and (5) become

$$(D_a f)(x) = \sqrt{|a|}\, f(ax), \qquad (T_b f)(x) = f(x-b), \quad x \in \mathbb{R}, \quad f \in L^2(\mathbb{R}),$$
$$T_{ab} = D_{a^{-1}} T_b D_a, \qquad a, b \in \mathbb{R}\backslash\{0\}. \tag{9}$$

In this context we have the following result.

Proposition 2.1. *Two pairs* (D_a, T_b), $(D_{a'}, T_{b'})$ (*where* $a, b, a', b' \in \mathbb{R}\backslash\{0\}$) *are equivalent if and only if* $a = a'$.

Proof. First assume that $a = a'$. Then, with $c = b'b^{-1}$, we have

$$(D_c)^*\, T_b D_c = D_{c^{-1}} T_b D_\delta = T_{cb} = T_{b'},$$
$$(D_c)^*\, D_a D_c = D_a.$$

Hence we have verified that $(D_a, T_b) \sim (D_{a'}, T_{b'})$.

For the necessity, let us assume that U is a unitary operator which arises from the equivalence $(D_a, T_b) \sim (D_{a'}, T_{b'})$ as in (7) and (8). According to what we just proved, (D_a, T_b) is equivalent to (D_a, T_1) and also $(D_{a'}, T_{b'})$ is equivalent to $(D_{a'}, T_1)$. Therefore (D_a, T_1) is equivalent to $(D_{a'}, T_1)$ via the unitary operator $V = D_b U D_{1/b'}$. Indeed,

$$V^* D_a V = D^*_{1/b'}\, U^* D^*_b\, D_a D_b\, U D_{1/b'}$$
$$= D^*_{1/b'}\, U^* D_a\, U D_{1/b'} = D^*_{1/b'}\, D_{a'} D_{1/b'} = D_{a'}, \tag{10}$$

and using (9), we obtain

$$V^* T_1 V = D^*_{1/b'}\, U^* D^*_b\, T_1 D_b\, U D_{1/b'}$$
$$= D^*_{1/b'}\, U^* T_b\, U D_{1/b'} = D^*_{1/'}\, T_{b'} D_{1/b'} = T_1. \tag{11}$$

So we can assume that $b = b' = 1$, and for brevity we simply write T in place of T_1. From (11), $TV = VT$ and thus $T^k V = VT^k$ for every integer k. (Note that $T^k = T_k$.) From (9) we also get

$$T_{na-m} = D_{a^{-1}} T^n D_a T^{-m}, \qquad m, n \in \mathbb{Z}. \tag{12}$$

We need two lemmas whose proofs will be given after we complete the proof of Proposition 2.1.

Lemma 2.2. *For any nonzero real number a, there exist sequences* $\{n_k\}$, $\{m_k\}$ *of integers such that*

(i) $n_k a - m_k \to 0$, *and*

(ii) $|n_k| \to \infty$.

LEMMA 2.3. *For every sequence $\{a_n\}$ of real numbers, $T_{a_n} f \xrightarrow{L^2} f$ for all $f \in L^2(\mathbb{R})$ if and only if $a_n \to 0$.*

To continue the proof of Proposition 2.1, we apply Lemma 2.2. for a and (12) for the integers n_k, m_k given by this lemma to get

$$T_{n_k a - m_k} = D_{a^{-1}} T^{n_k} D_a T^{-n_k}, \qquad k \in \mathbb{N}.$$

Conjugating the above equality with V we obtain

$$\begin{aligned} V^* T_{n_k a - m_k} V &= D_{a'^{-1}} T^{n_k} D_{a'} T^{-m_k} = T_{n_k a' - m_k}, \\ T_{n_k a' - m_k} &= V^* T_{n_k a - m_k} V. \end{aligned} \tag{13}$$

Now we apply Lemma 2.3. Since $n_k a - m_k \to 0$, $T_{n_k a - m_k} f \xrightarrow{L^2} f$ for all $f \in L^2(\mathbb{R})$, and also

$$V^* T_{n_k a - m_k} V f \xrightarrow{L^2} f, \qquad f \in L^2(\mathbb{R}).$$

Therefore by (13), $T_{n_k a' - m_k} f \xrightarrow{L^2} f$ for all $f \in L^2(\mathbb{R})$, and using Lemma 2.3 again we obtain that $n_k a' - m_k \to 0$. Hence $a' = \lim(m_k/n_k) = a$, and Proposition 2.1 is proved. ∎

Proof of Lemma 2.2. Of course, we may suppose that $a > 0$. If a is a rational number, say $a = p/q$, with $p, q \in \mathbb{Z}$, and $q \neq 0$, we can choose $m_k = kp$ and $n_k = kq$. If a is an irrational number, consider the sequence $\{\varepsilon_k\}$ of decimal parts, $\lfloor ka \rfloor$, of the sequence $\{ka\}_{k \in \mathbb{N}}$. Then the sequence $\{\varepsilon_k\}$, being a sequence of distinct numbers in $[0, 1]$, has a cluster point. This implies that for any given $\eta > 0$, there exist sufficiently large integers j_1, j_2 such that $|\varepsilon_{j_1} - \varepsilon_{j_2}| < \eta$. That is, $|j_1 a - p_1 - (j_2 a - p_2)| < \eta$ for certain integers p_1, p_2. Rewriting this we obtain that

$$|(j_1 - j_2)\, a - (p_1 - p_2)| < \eta$$

Taking $\eta_k = 1/k$, $k \in \mathbb{N}$, we define the required sequences $\{n_k\}$, $\{m_k\}$ by $n_k = j_1(\eta_k) - j_2(\eta_k)$, $m_k = p_1(\eta_k) - p_2(\eta_k)$. We can also choose $j_1(\eta_k)$, $j_2(\eta_k)$ in such a way that $|n_k| \to \infty$. ∎

Proof of Lemma 2.3. Assume $a_n \to 0$. Then $T_{a_n} f \xrightarrow{L^2} f$ is equivalent to

$$\int_{\mathbb{R}} |f(x - a_n) - f(x)|^2 \, dx \to 0.$$

This is easily seen to be true if f is a continuous function with compact support by an argument using uniform continuity. Since the T_{a_n} are unitary operators and the continuous functions with compact support are dense in $L^2(\mathbb{R})$, $T_{a_n} f \xrightarrow{L^2} f$ for all $f \in L^2(\mathbb{R})$.

For the necessity, suppose that $\{a_n\}$ does not converge to zero. Then there exists an $\varepsilon_0 > 0$ and a subsequence $\{a_{n_k}\}_k$ such that $|a_{n_k}| \geqslant \varepsilon_0$. Taking $f = \chi_{[0, \varepsilon_0/2]}$ we have

$$\|T_{a_{n_k}} f - f\|^2 = \int_{\mathbb{R}} |f(x - a_{n_k}) - f(x)|^2 \, dx$$

$$= \int_{\mathbb{R}} |\chi_{[0, \varepsilon_0/2]}(x - a_{n_k}) - \chi_{[0, \varepsilon_0/2]}(x)|^2 \, dx = \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0,$$

which contradicts the assumption that $T_{a_n} f \xrightarrow{L^2} f$ for all $f \in L^2(\mathbb{R})$. ∎

3. THE CASE $n \geqslant 2$

We have the following generalization of Proposition 2.1. It shows that the only way two $(n+1)$-tuples can be equivalent is the "natural" way.

THEOREM 3.1. *Two $(n+1)$-tuples (D_A, T_{v_i}) and (D_B, T_{w_i}) are equivalent if and only if $B = C^{-1}AC$, where C is the matrix of the linear transformation $\tilde{C}$ defined on the basis $\{w_1, ..., w_n\}$ by $\tilde{C}w_i = v_i$, $i = 1, ..., n$, written relative to the canonical basis for $\mathbb{R}^n$.*

Proof. Setting $U = D_{C^{-1}}$, we easily calculate that (7) is satisfied, and using (5) with $A = C^{-1}$ and $v = v_i$, we get (8). This proves the sufficiency of the given conditions, so we turn to the necessity. Suppose (7) and (8) are satisfied by a unitary operator U. We write $T_i = T_{e_i}$ where $\{e_1, e_2, ..., e_n\}$ is the canonical basis for $\mathbb{R}^n$. One can easily check that (D_A, T_{v_i}) is equivalent to $(D_{C_1^{-1}AC_1}, T_i)$, where C_1 is the matrix of the linear transformation $\tilde{C}_1$ defined by $\tilde{C}_1 e_i = v_i$, $i = 1, ..., n$, written relative to the ordered basis $\{e_i\}$. Similarly (D_B, T_{w_i}) is equivalent to $(D_{C_2^{-1}BC_2}, T_i)$, where $\tilde{C}_2 e_i = w_i$, $i = 1, ..., n$. Putting these facts together and writing $V = D_{C_2} U D_{C_1^{-1}}$, we obtain that $(D_{C_1^{-1}AC_1}, T_i)$ is equivalent to $(D_{C_2^{-1}BC_2}, T_i)$, as the following calculation shows:

$$V^* D_{C_2^{-1}BC_2} V = D_{C_1} U^* D_{C_2^{-1}} D_{C_2^{-1}BC_2} D_{C_2} U D_{C_1^{-1}}$$

$$= D_{C_1} U^* D_B U D_{C_1^{-1}} = D_{C_1} D_A D_{C_1^{-1}} = D_{C_1^{-1}AC_1}.$$

Using (5) repeatedly, we get

$$\begin{aligned} V^*T_iV &= D_{C_1}U^*D_{C_2^{-1}}T_iD_{C_2}UD_{C_1^{-1}} = D_{C_1}U^*T_{\tilde{C}_2e_i}UD_{C_1^{-1}} \\ &= D_{C_1}U^*T_{w_i}UD_{C_1^{-1}} = D_{C_1}T_{v_i}D_{C_1^{-1}} \\ &= T_{\tilde{C}_1^{-1}v_i} = T_{e_i} = T_i, \qquad i = 1, ..., n. \end{aligned} \tag{14}$$

In other words, if we write $A' = C_1^{-1}AC_1$ and $B' = C_2^{-1}BC_2$, it suffices to show that $A' = B'$, since this implies that

$$B = C_2C_1^{-1}AC_1C_2^{-1} = (C_1C_2^{-1})^{-1}A(C_1C_2^{-1}),$$

and if $C = C_1C_2^{-1}$, then $\tilde{C}w_i = \tilde{C}_1\tilde{C}_2^{-1}w_i = \tilde{C}_1e_i = v_i$, $i = 1, ..., n$, as was to be proved.

For the purpose of showing that $A' = B'$, we need two lemmas whose proofs will follow the end of the proof of Theorem 3.1.

LEMMA 3.2. *For a sequence of vectors* $\{u_k\} \subset \mathbb{R}^n$, $T_{u_k}f \xrightarrow{L^2} f$ *for all* $f \in L^2(\mathbb{R}^n)$ *if and only if* $\lim_k \|u_k\| = 0$.

LEMMA 3.3. *If* $u \in \mathbb{R}^n \setminus \{0\}$, *then there exist sequences of integers* $\{m_k^0\}_k$, $\{m_k^1\}_k$, ..., $\{m_k^n\}_k$ *such that*

(a) $\lim_k \|m_k^0u - \sum_{i=1}^n m_k^ie_i\| = 0$, *and*

(b) $\lim_k |m_k^0| = +\infty$.

To continue the proof of Theorem 3.1, note that (5) with $v = m_k^0A'e_1$ and $A = (A')^{-1}$ gives

$$T_{m_k^0A'e_1} = D_{A'}^*T_1^{m_k^0}D_{A'},$$

and from this one concludes easily that

$$T_{m_k^0A'e_1 - \sum_{i=1}^n m_k^ie_i} = D_{A'}^*T_1^{m_k^0}D_{A'}T_1^{-m_k^1}T_2^{-m_k^2}\cdots T_n^{-m_k^n}. \tag{15}$$

Applying Lemma 3.3 to $u = A'e_1$, we obtain the existence of the corresponding sequences of integers $\{m_k^0\}_k$, ..., $\{m_k^n\}_k$ having the properties (a) and (b). Conjugating (15) with V, and taking into account (14), we obtain

$$VT_{m_k^0A'e_1 - \sum_{i=1}^m m_k^ie_i}V^* = T_{m_k^0B'e_1 - \sum_{i=1}^n m_k^ie_i}, \tag{16}$$

and using (a) we may apply Lemma 3.2 to (16) and obtain that

$$\lim_{k} \left\| m_k^0 B' e_1 - \sum_{i=1}^{n} m_i^k e_i \right\| = 0.$$

Therefore $B'e_1 = \lim_{k\to\infty} (\sum_{i=1}^{n} m_k^i e_i)/m_k^0 = A'e_1$, and clearly the same argument shows that $B'e_i = A'e_i$, $i = 2, ..., n$. This proves that $A' = B'$, and completes the proof of Theorem 3.1. ▮

Proof of Lemma 3.2. Suppose $\lim_k u_k = 0$. Then, for each $f \in L^2(\mathbb{R}^n)$, the condition $T_{u_k} f \xrightarrow{L^2} f$ is equivalent to

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |f(x-u_k) - f(x)|^2 \, dx = 0. \tag{17}$$

This is easy to check for continuous functions with compact support using uniform continuity. Since $\|T_{u_k}\| \leqslant 1$ for all k and since such functions are dense in $L^2(\mathbb{R}^n)$, (17) holds for all $f \in L^2(\mathbb{R}^2)$.

Suppose now that $u_k \nrightarrow 0$. Then there exists an $\varepsilon_0 > 0$ and a subsequence $\{u_{k_\ell}\}$ of $\{u_k\}$ satisfying $\|u_{k_\ell}\| \geqslant \varepsilon_0$ for all $\ell \in \mathbb{N}$. Taking $f = \chi_{B(0,\, \varepsilon_0/2)}$, where $B(0, \varepsilon_0/2)$ is the open ball in $\mathbb{R}^n$ centered at 0 and having radius $\varepsilon_0/2$, we obtain

$$\int_{\mathbb{R}^n} |f(x-u_{k_\ell}) - f(x)|^2 \, dx = 2\mu_n(B(0, \varepsilon_0/2)), \qquad \ell \in \mathbb{N}. \tag{18}$$

Thus $T_{u_k} f \nrightarrow f$, and the lemma is proved. ▮

Proof of Lemma 3.3. *There exist real numbers* $\{c_1, c_2, ..., c_n\}$, *not all zero, such that*

$$u = \sum_{i=1}^{n} c_i e_i.$$

For any positive integer p, we consider the vector $v_p = \sum_{i=1}^{n} \lfloor pc_i \rfloor e_i$ where by $\lfloor x \rfloor$ we denote, as before, the decimal part of the real number x. Since $\|v_p\| \leqslant \sqrt{n}$, $p \in \mathbb{N}$, and the closed balls in $\mathbb{R}^n$ are compact sets, given $\varepsilon_k = 1/k$, $k \in \mathbb{N}$, there exists positive integers $p_1(k)$, $p_2(k)$ such that $|p_1(k) - p_2(k)| > k$ and $\|v_{p_1(k)} - v_{p_2(k)}\| < \varepsilon_k$. In other words,

$$\left\| (p_1(k) - p_2(k))\, u - \sum_{i=1}^{n} ([p_1(k)\, c_i] - [p_2(k)\, c_i])\, e_i \right\|$$
$$= \|v_{p_1(k)} - v_{p_2(k)}\| < \varepsilon_k,$$

where $[x]$ is the greatest integer smaller than x (i.e., $x=[x]+\lfloor x \rfloor$). Hence, writing $m_k^0 = p_1(k) - p_2(k)$, $m_k^i = ([p_1(k)\, c_i] - [p_2(k)\, c_i])$, $i = 1, ..., n$, we see that (a) and (b) are satisfied. ▮

4. BILATERAL SHIFTS

Note that if $A \in M_n'(\mathbb{R})$, $\{v_n, ..., v_n\}$ is a basis for $\mathbb{R}^n$, and $\mathscr{U}_{D_A, T_{v_i}}$ admits a wavelet ψ, then D_A must be a bilateral shift of infinite multiplicity. Indeed, an infinite-dimensional complete wandering subspace for D_A will be $E_\psi := \bigvee \{T_{v_1}^{\ell_1} T_{v_2}^{\ell_2} \cdots T_{v_n}^{\ell_n} \psi : \ell_1, ..., \ell_n \in \mathbb{Z}\}$. In particular, if $A \in M_n'(\mathbb{R})$ and either A or A^{-1} is expansive, then D_A is a bilateral shift of infinite multiplicity because $\mathscr{U}_{D_A, T_{v_i}}$ admits a wavelet [6]. Another way of seeing this in case A^{-1} is expansive is that if we set $\mathscr{F} = \mathscr{B} \backslash A\mathscr{B}$, where $\mathscr{B}$ is the closed unit ball of $\mathbb{R}^n$, then $\{A^\ell \mathscr{F} : \ell \in \mathbb{Z}\}$ is a measurable partition of $\mathbb{R}^n$. Thus $L^2(\mathscr{F})$, considered as a closed subspace of $L^2(\mathbb{R}^n)$, is an infinite dimensional complete wandering subspace for D_A. If $v \in \mathbb{R}^n \backslash \{0\}$, then T_v is always a bilateral shift of infinite multiplicity. Indeed, T_v is unitarily equivalent to T_{e_1}, and a complete wandering subspace for T_{e_1} is $L^2(G) \subseteq L^2(\mathbb{R}^n)$ where $G = [0, 1] \times \mathbb{R}^{(n-1)}$. There are many $A \in M_n'(\mathbb{R})$ with neither A nor A^{-1} expansive such that D_A is a bilateral shift of infinite multiplicity (cf. Theorem 4.2 below). For such an A, with $\{v_1, ..., v_n\}$ a basis for $\mathbb{R}^n$, it is possible (we have neither an example nor a counterexample) that the unitary system $\mathscr{U}_{A, T_{v_i}}$ admits a wavelet. Let us call such wavelets, if they exist, "non-standard wavelets."

Problem 4.1. Do non-standard wavelets exist?

In view of Question 4.1, it is of interest to know precisely which matrices $A \in M_n'(\mathbb{R})$ lead to bilateral shifts D_A. We give a simple criterion for this.

THEOREM 4.2. *Let $A \in M_n'(\mathbb{R})$. Then D_A is a bilateral shift of infinite multiplicity if and only if A is not similar (in $M_n(\mathbb{C})$) to a unitary matrix.*

Proof. By way of contradiction, assume that there are $S, U \in M_n'(\mathbb{C})$ with U unitary such that $SAS^{-1} = U$. An argument like the one used in the proof of Lemma 2.2 or Lemma 3.3 shows that there is a sequence of integers $m_k \to \infty$ such that

$$U^{m_k} x \to x, \qquad x \in \mathbb{C}^n. \tag{19}$$

This implies that

$$A^{m_k} x = S^{-1} U^{m_k} S x \to S^{-1} S x = x, \qquad x \in \mathbb{R}^n.$$

Since $(D_A)^p = D_{A^p}$, we obtain that

$$\|D_A^{m_k} f - f\|^2 = \int_{\mathbb{R}^n} |f(A^{m_k}x) - f(x)|^2 \, d\lambda(x) \to 0, \qquad f \in L^2(\mathbb{R}^n), \quad (20)$$

by an argument using the density in $L^2(\mathbb{R}^n)$ of the continuous functions with compact support and the remark that (19) is uniform on compact sets of $\mathbb{R}^n$. From (20) we obtain that

$$\begin{aligned} &\|D_A^{m_k} f\|^2 + \|f\|^2 - 2\,\mathrm{Re}\langle D_A^{m_k} f, f\rangle \\ &\quad = 2\,\|f\|^2 - 2\,\mathrm{Re}\langle D_A^{m_k} f, f\rangle \to 0, \qquad f \in L^2(\mathbb{R}^n). \end{aligned} \quad (21)$$

Since D_A is a bilateral shift, and it is well-known that the sequence of positive powers of such a shift converges to zero in the weak operator topology, the term $2\,\mathrm{Re}\langle D_A^{m_k} f, f\rangle$ in (21) tends to 0 for each $f \in L^2(\mathbb{R}^n)$, and this clearly contradicts (21).

Going the other way, suppose now that A is not similar (in $M_n(\mathbb{C})$) to a unitary matrix. To show that D_A is a bilateral shift of infinite multiplicity, it suffices to exhibit a Borel set $\mathscr{B} \subset \mathbb{R}^n$ such that $\mathbb{R}^n$ is the disjoint union $\bigcup_{k \in \mathbb{Z}} A^k \mathscr{B}$ modulo a set of μ_n measure zero. (Indeed, it follows easily from this that $L^2(\mathscr{B})$ $(\subset L^2(\mathbb{R}^n))$ is an infinite dimensional wandering subspace for D_A such that $\bigoplus_{k \in \mathbb{Z}} D_A^k L^2(\mathscr{B}) = L^2(\mathbb{R}^n)$). To construct such a set $\mathscr{B}$, we need a result on Borel selection whose proof will follow the completion of the proof of Theorem 4.2.

Proposition 4.3. *Suppose $A \in M'_n(\mathbb{R})$ and $\mathscr{S}$ is a proper linear manifold in $\mathbb{R}^n$ invariant under A such that for all x in $\mathbb{R}^n \backslash \mathscr{S}$, the accumulation points of the set $\{A^k x : k \in \mathbb{Z}\}$ lie in $\mathscr{S}$. Let $\mathfrak{R}$ be the equivalence relation on $\mathbb{R}^n \backslash \mathscr{S}$ defined by $x \mathfrak{R} y$ if and only if there exists an integer k with $A^k x = y$. Then there exists a Borel set $\mathscr{B} \subset \mathbb{R}^n \backslash \mathscr{S}$ such that $\mathscr{B}$ meets every equivalence class of the relation $\mathfrak{R}$ in a singleton.*

Suppose, for the moment, that we have specified such a linear manifold $\mathscr{S} \subset \mathbb{R}^n$, and let $\mathscr{B}$ be a Borel set given by Proposition 4.3. Then $\mathbb{R}^n \backslash \mathscr{S}$ is the disjoint union $\bigcup_{k \in \mathbb{Z}} A^k \mathscr{B}$, and thus D_A is a bilateral shift of infinite multiplicity. Thus to complete the proof of Theorem 4.2, it suffices to exhibit a proper linear manifold $\mathscr{S} \subset \mathbb{R}^n$ with the appropiate properties. There are two cases to consider.

Case I. Each eigenvalue of A has modulus one. In this case, regarding A as an operator on $\mathbb{C}^n$ in the usual fashion, we know that there exists an invariant subspace $\mathscr{M} \subset \mathbb{C}^n$ for A such that the Jordan matrix corresponding to $A_{|\mathscr{M}}$ is a single, nondiagonal Jordan block associated with some eigenvalue λ_0 of A. Hence $\mathscr{M}$ has a complementary subspace $\mathscr{N}$ (i.e.,

$\mathscr{M} + \mathscr{N} = \mathbb{C}^n$) that is also invariant under A. Let $x_0 \in \mathscr{M}$ be an eigenvector for A, and let P be the idempotent (commuting with A) such that *range* $P = \mathscr{M}$ and *ker* $P = \mathscr{N}$. We next define the proper subspace $\mathscr{T}$ of $\mathbb{C}^n$ by

$$\mathscr{T} = \left\{ x \in \mathbb{C}^n : Px \in \bigvee \{x_0\} \right\}.$$

Note that if $z \in \mathscr{T}$, then $Pz = \beta x_0$ for some $\beta \in \mathbb{C}$ and hence $PAz = APz = \beta Ax_0 = \beta\lambda_0 x_0$. Thus $\mathscr{T}$ is invariant for A. We shall show that if $y \in \mathbb{C}^n \backslash \mathscr{T}$, then the sequences $\{\|A^n y\|\}_{n=1}^{\infty}$ and $\{\|A^{-n} y\|\}_{n=1}^{\infty}$ converge to $+\infty$, and thus the set $\{A^n y : n \in \mathbb{Z}\}$ has no points of accumulation. Assuming for the moment that this has been established, we define the proper subspace $\mathscr{S}$ of $\mathbb{R}^n$ by $\mathscr{S} = \mathbb{R}^n \cap \mathscr{T}$. Clearly $\mathscr{S}$ is invariant under A, and since $\mathbb{R}^n \backslash \mathscr{S} \subset \mathbb{C}^n \backslash \mathscr{T}$, for each x in $\mathbb{R}^n \backslash \mathscr{S}$, $\{A^n x : n \in \mathbb{Z}\}$ has no point of accumulation, and thus $\mathscr{S}$ has the appropriate properties to make Proposition 4.2 applicable. To see that for each $y \in \mathbb{C}^n \backslash \mathscr{T}$, $\lim_{|n| \to \infty} \|A^n y\| = +\infty$, it suffices to show that $\lim_{|n| \to \infty} \|A^n Py\| = +\infty$ (since $A\mathscr{N} \subset \mathscr{N}$). But $A^n(Py) = (A_{|\mathscr{M}})^n (Py)$, $n \in \mathbb{Z}$, and the powers of $A_{|\mathscr{M}}$ behave as the powers of a Jordan block matrix. The result now follows from an elementary computation which we omit. Thus the proof of Case I is complete.

Case II. Some (real or complex) eigenvalue λ_0 of A satisfies $|\lambda_0| \neq 1$. The cases $|\lambda_0| < 1$ and $|\lambda_0| > 1$ are entirely similar, so we treat only the case $|\lambda_0| > 1$. Moreover, the argument is much like that of Case I with small changes, so we sketch only the necessary changes. First, let $\mathscr{M} \subset \mathbb{C}^n$ be an invariant subspace for A with the properties that the Jordan form for the operator $A_{|\mathscr{M}}$ is a single Jordan block associated with the eigenvalue λ_0. Then $\mathscr{M}$ has a complement $\mathscr{N}$ that is invariant under A, and we define P as in Case I. However we define

$$\mathscr{T} = \{x \in \mathbb{C}^n : Px = 0\}.$$

As before, $\mathscr{T} \neq \mathbb{C}^n$, $PA = AP$, $A\mathscr{T} \subset \mathscr{T}$, and we take $\mathscr{S} = \mathbb{R}^n \cap \mathscr{T}$. Then $A\mathscr{S} \subset \mathscr{S}$, and to show that for each $x \in \mathbb{R}^n \backslash S$, all accumulation points of the set $\{A^n x : n \in \mathbb{Z}\}$ lie in $\mathscr{S}$, it suffices to establish that for all vectors y in $\mathbb{C}^n \backslash \mathscr{T}$, the set $\{A^n y : n \in \mathbb{Z}\}$ has accumulation points only in $\mathscr{T}$. In fact, as an easy calculation (using $|\lambda_0| > 1$ and $A\mathscr{N} \subset \mathscr{N}$) shows, for such a vector y, $\{\|A^n y\|\}_{n=1}^{\infty}$ converges to $+\infty$ and $\{\|A^{-n} Py\|\}_{n=1}^{\infty}$ tends to zero. Thus every accumulation point z_0 of the set $\{A^n y : n \in \mathbb{Z}\}$ satisfies $Pz_0 = \lim_k PA^{-n_k} y = \lim_k A^{-n_k} Py = 0$ for some subsequence $\{n_k\}$ of $\mathbb{N}$, and hence $z_0 \in \mathscr{T}$. Thus the proof of Theorem 4.2 is complete.

Proof of Proposition 4.3. We use the following principle of Borel selection [2, p. 206]: Suppose X is a nonempty complete separable metric space, and let R be an equivalence relation on X such that the equivalence classes mod R are closed sets in X and such that for each closed subset F of X, the set $R_F \subset X$ consisting of all elements of X that are R-related to some element of F is a Borel set in X. Then there exists a Borel set $\mathscr{B}$ in X such that $\mathscr{B}$ meets every equivalence class mod R in a singleton.

We show that Proposition 4.3 follows from this principle. First note that since $\mathscr{S}$ is a proper subspace of $\mathbb{R}^n$, $\mathbb{R}^n \backslash \mathscr{S}$ is a nonempty open set in $\mathbb{R}^n$. Thus (cf. [3, Problem 8D]) there exists an equivalent metric ρ on $\mathbb{R}^n \backslash \mathscr{S}$ that turns $\mathbb{R}^n \backslash \mathscr{S}$ into a complete metric space. Since for each x in $\mathbb{R}^n \backslash \mathscr{S}$, all the accumulation points (in either metric) of the set $\{A^n x : n \in \mathbb{Z}\}$ lie in $\mathscr{S}$, every equivalence class in $(\mathbb{R}^n \backslash \mathscr{S}, \rho)$ corresponding to the relation $R = \mathfrak{R}$ is closed. Moreover, if F is any closed subset of $(\mathbb{R}^n \backslash \mathscr{S}, \rho)$, then R_F (defined above) equals $\bigcup_{n \in \mathbb{Z}} A^n F$, and hence is an F_σ in $(\mathbb{R}^n \backslash \mathscr{S}, \rho)$. Hence there exists a Borel subset $\mathscr{B}$ of $\mathbb{R}^n \backslash \mathscr{S}$ (relative to ρ) that meets each equivalence class mod $\mathfrak{R}$ in a singleton, and obviously $\mathscr{B}$ is also a Borel set in $\mathbb{R}^n$ under its Euclidian metric. ∎

5. WEAK EQUIVALENCE OF UNITARY SYSTEMS

Thus far we have discussed equivalence of $(n+1)$-tuples (D_A, T_{v_i}). There is a less restrictive notion of equivalence that was considered in [5] and by other authors: unitary systems $\mathscr{U}_{D_A, T_{v_i}}$ and $\mathscr{U}_{D_B, T_{w_i}}$ acting on the Hilbert space $L^2(\mathbb{R}^n)$ are said to be weakly equivalent (in [5], unitarily equivalent) if $(D_A, T_{v_i}) \approx (D_B, T_{w_i})$ (cf. Definition 1.2). In this case $V\mathscr{W}_{A, v_i} = \mathscr{W}_{B, w_i}$, so the sets of vectors $\mathscr{W}_{A, v_i}$ and $\mathscr{W}_{B, w_i}$ have the same topological and structural properties. It can happen that (D_A, T_{v_i}) and (D_B, T_{w_i}) are not equivalent and yet the unitary systems $\mathscr{U}_{D_A, T_{v_i}}$ and $\mathscr{U}_{D_B, T_{w_i}}$ are weakly equivalent (in fact they can be equal). For instance, this happens when A is expansive, $B = A^{-1}$, and $\{v_i\}$, $\{w_i\}$ generate the same additive subgroup of $\mathbb{R}^n$. The purpose of this section is to establish the connections between equivalence of two such $(n+1)$-tuples and weak equivalence of the unitary systems they generate. We will show (Theorem 5.8 and Corollary 5.9) that if $\{v_1, ..., v_n\}$ and $\{w_1, ..., w_n\}$ are bases of $\mathbb{R}^n$ and if A, B are *expansive* matrices in $M'_n(\mathbb{R})$, (or, more generally, if A or B has property $\mathscr{P}$ of Definition 5.6), then the systems $\mathscr{U}_{D_A, T_{v_i}}$ and $\mathscr{U}_{D_B, T_{w_i}}$ are weakly equivalent if and only if there is an equivalent pair of $(n+1)$-tuples which generate the corresponding systems.

We require several technical lemmas which concern elementary properties of the operators T_x, and D_A studied above.

LEMMA 5.1. *Let $B \in M'_n(\mathbb{R})$ and $x \in \mathbb{R}^n$. Then for $p \in \mathbb{N}$,*

(i) $(D_B T_x)^p = D_B^p T_{(I + B + B^2 + \cdots + B^{p-1})x}$, *and*

(ii) $(D_B T_x)^{-p} = D_B^{-p} T_{-(B^{-1} + B^{-2} + \cdots + B^{-p})x}$.

Proof. Using (12) a trivial calculation gives $(D_B T_x)^2 = D_B^2 T_{(I+B)\,x}$. Now assume, by induction, that (i) is valid for some $p > 2$. Then

$$\begin{aligned}(D_B T_x)^{p+1} &= D_B T_x D_B^p T_{(I+\,\cdots\,+B^{p-1})\,x}\\ &= D_B D_B^p T_{B^p x} T_{(I+\,\cdots\,+B^{p-1})\,x}\\ &= D_B^{p+1} T_{(I+\,\cdots\,+B^p)\,x}\end{aligned}$$

as required, proving (i).

As for (ii), we have $(D_B T_x)^{-1} = T_x^{-1} D_B^{-1} = T_{-x} D_{B^{-1}} = D_{B^{-1}} T_{B^{-1}(-x)}$. Thus by (i),

$$\begin{aligned}(D_B T_x)^{-p} &= (D_{B^{-1}} T_{-B^{-1}x})^p\\ &= D_{B^{-1}}^p T_{(I+B^{-1}+B^{-2}+\,\cdots\,+B^{-(p-1)})(-B^{-1}x)}\\ &= D_B^{-p} T_{-(B^{-1}+\,\cdots\,+B^{-p})\,x}. \quad \blacksquare\end{aligned}$$

LEMMA 5.2. *Suppose $A \in M_n'(\mathbb{R})$ and $x \in \mathbb{R}^n$. Then*

(i) $D_A T_x = T_x D_A$ *if and only if* $Ax = x$, *and*

(ii) $D_A = T_x$ *if and only if* $A = I$ *and* $x = 0$.

Proof. Since $T_x D_A = D_A T_{Ax}$, the condition $D_A T_x = T_x D_A$ is equivalent to $T_{Ax} = T_x$, which is in turn equivalent to $Ax = x$, which proves (i).

If $D_A = T_x$, then D_A commutes with T_y for all $y \in \mathbf{R}^n$. Thus (i) implies that $A = I$ and $D_A = I = T_x$, which gives $x = 0$. $\blacksquare$

LEMMA 5.3. *If $A_1, A_2 \in M_n'(\mathbb{R})$ and $x_1, x_2 \in \mathbb{R}^n$, then $D_{A_1} T_{x_1} = D_{A_2} T_{x_2}$ if and only if $A_1 = A_2$ and $x_1 = x_2$.*

Proof. Only one direction needs proof. If $D_{A_1} T_{x_1} = D_{A_2} T_{x_2}$, then $D_{A^{-1}} D_{A_2} = D_{A_2 A_1^{-1}} = T_{x_1}(T_{x_2})^{-1} = T_{x_1 - x_2}$. Thus $A_1^{-1} A_2 = I$ and $x_1 - x_2 = 0$ by Lemma 5.2 (ii). $\blacksquare$

If $\mathscr{S}$ is a set of vectors in $\mathbb{R}^n$ we will write $\mathrm{span}_{\mathbb{Z}}\,\mathscr{S}$ for the additive subgroup of $\mathbb{R}^n$ generated by $\mathscr{S}$. (Equivalently, $\mathrm{span}_{\mathbb{Z}}\,\mathscr{S}$ is the family of all linear combinations of vectors in $\mathscr{S}$ with integer coefficients.) In particular, if $\mathscr{S} = \{v_1, v_2, ..., v_n\}$ we write $\mathrm{span}_{\mathbb{Z}}\{v_i\}$ for $\mathrm{span}_{\mathbb{Z}}\,\mathscr{S}$.

LEMMA 5.4. *Let $A \in M_n'(\mathbb{R})$, let $\{v_1, ..., v_n\}$ be a basis for $\mathbb{R}^n$, and let* $\mathscr{U} = \mathscr{U}_{D_A, T_{v_i}}$. Then

(i) $T_x \in \mathscr{U}$ *if and only if* $x \in \mathrm{span}_{\mathbb{Z}}\{v_i\}$,

(ii) $T_x \mathscr{U} \subseteq \mathscr{U}$ *if and only if* $A^p x \in \mathrm{span}_{\mathbb{Z}}\{v_i\}$ *for all* $p \in \mathbb{Z}$, *and*

(iii) *If* $B \in \mathcal{U}$, *then* $B^p \in \mathcal{U}$ *for all* $p \in \mathbb{Z}$ *if and only if* $B = D_A^q T_x$ *for some* $q \in \mathbb{Z}$ *and* $x \in \mathbb{R}$ *with the property that* $A^{pq}x \in \operatorname{span}_{\mathbb{Z}}\{v_i\}$ *for all* $p \in \mathbb{Z}$.

Proof. If $T_x \in \mathcal{U}$, we write $T_x = D_A^q T_{v_1}^{r_1} \cdots T_{v_n}^{r_n}$ for some $(r_1, ..., r_n) \in \mathbb{Z}^n$. Then $D_A^q = T_z$ for $z = x - (r_1 v_1 + \cdots + r_n v_n)$. So by Lemma 5.2 (ii), $z = 0$ and $A^q = I$, which proves the necessity in (i); the other implication in (i) is trivial.

If $T_x \mathcal{U} \subseteq \mathcal{U}$, then $T_x D_{A^p} = D_{A^p} T_{A^p x} \in \mathcal{U}$ for all $p \in \mathbb{Z}$. Thus $T_{A^p x} \in \mathcal{U}$ for every $p \in \mathbb{Z}$, and so $A^p x \in \operatorname{span}_{\mathbb{Z}}\{v_i\}$ for all $p \in \mathbb{Z}$ by (i). The other implication of (ii) is clear from similar computations.

Suppose now that $B^p \in \mathcal{U}$ for all $p \in \mathbb{Z}$. Since $B \in \mathcal{U}$, $B = D_A^q T_x = D_{A^q} T_x$ for some $q \in \mathbb{Z}$ and $x \in \operatorname{span}_{\mathbb{Z}}\{v_1, ..., v_n\}$. A short calculation using Lemma 5.1 yields that $T_{A^{pq}x} \in \mathcal{U}$ for all $p \in \mathbb{Z}$. Therefore x has the required property by (ii). A similar calculation proves the other implication of (iii). ∎

Proposition 5.5. *Suppose* $C \in M_n'(\mathbb{R})$ *and* $\{v_1, ..., v_n\}$ *is a basis for* $\mathbb{R}^n$. *For fixed* $x \in \mathbb{R}^n \setminus \{0\}$ *the following are equivalent*:

(i) $C^p x \in \operatorname{span}_{\mathbb{Z}}\{v_i\}$ *for all* $p \in \mathbb{Z}$,

(ii) $C^p x \in \operatorname{span}_{\mathbb{Z}}\{v_i\}$ *for* $p = 0, ..., (d-1)$, *where d is the dimension of the cyclic invariant subspace* $\mathcal{S}_x = \bigvee \{x, Cx, ..., C^{d-1}x\}$ *for the operator C on* $\mathbb{R}^n$, *and the characteristic polynomial* χ *of the restriction operator* $C|_{\mathcal{S}_x}$ *has the form*

$$\chi(t) = t^d + a_1 t^{d-1} + \cdots + a_{d-1} t + a_d, \tag{22}$$

where all the coefficients a_j *are integers and* $a_d = \pm 1$.

Proof. We prove the implication (ii) ⇒ (i) by mathematical induction using the fact that $\chi(C|_{\mathcal{S}_x}) = 0$. From (22) we get

$$\begin{aligned} C^d x &= C|_{\mathcal{S}_x}^d x = -a_1 C|_{\mathcal{S}_x}^{d-1} x - \cdots - a_d x \\ &= -a_1 C^{d-1} x - \cdots - a_d x \in \operatorname{span}_{\mathbb{Z}}\{v_1, ..., v_n\}. \end{aligned}$$

By induction, suppose that $p \in \mathbb{N} \setminus \{1, ..., d-1\}$ and the vectors $x, Cx, ..., C^p x$ lie in $\operatorname{span}_{\mathbb{Z}}\{v_i\}$. Then

$$\begin{aligned} C^{p+1} x &= (C|_{\mathcal{S}_x})^{p+1} x = -a_1 (C|_{\mathcal{S}_x})^p x - \cdots - a_d (C|_{\mathcal{S}_x})^{p-d+1} x \\ &= -a_1 C^p x - \cdots - a_d C^{p-d+1} x \in \operatorname{span}_{\mathbb{Z}}\{v_1, ..., v_n\}. \end{aligned}$$

For $-p \in \mathbb{N}$ one argues in a similar way using the hypothesis $a_d = \pm 1$.

To show that (i) implies (ii), we use the following facts about Gramians [12]. If $s \in \mathbb{N}$, L is an operator on an s-dimensional real Hilbert space $\mathcal{H}_s$

with inner product $\langle\ ,\ \rangle$, and $\{x_1, ..., x_s\}$, $\{y_1, ..., y_s\}$ are two sets of vectors in $\mathcal{H}_s$, we write $G(\{x_i\}, \{y_j\}) = \det(\langle x_i, y_j \rangle_{1 \leqslant i, j \leqslant s})$. One knows that $G(\{x_i\}, \{y_j\}) \neq 0$ if and only if $\{x_i\}$, $\{y_i\}$ are bases for $\mathcal{H}_s$, and moreover, when $G(\{x_i\}, \{y_j\}) \neq 0$,

$$\det(L) = \frac{G(\{Lx_i\}, \{y_j\})}{G(\{x_i\}, \{y_j\})}. \tag{23}$$

We apply these facts to the operator $L = (C|_{\mathcal{S}_x} - tI|_{\mathcal{S}_x})^k$ on $\mathcal{S}_x$, where $k \in \mathbb{N}$, $x_1 = y_1 = x$, $x_2 = y_2 = Cx$, ..., $x_d = y_d = C^{d-1}x$ $(s = d)$, $t \in \mathbb{R}$, and the inner product on $\mathcal{S}_x \subset \mathbb{R}^n$ is the restriction of the inner product $\langle \sum_{i=1}^d a_i v_i, \sum_{i=1}^d b_i v_i \rangle = \sum_{i=1}^d a_i b_i$ on $\mathbb{R}^n$. From (23) we obtain

$$\det((C|_{\mathcal{S}_x} - tI_{\mathcal{S}_x})^k) = \frac{G(\{(C-tI)^k x, ..., (C-tI)^k C^{d-1}x\}, \{x, ..., C^{d-1}x\})}{G(\{x, ..., C^{d-1}x\}, \{x, ..., C^{d-1}x\})}. \tag{24}$$

As one may easily observe using (i), $M = G(\{x, ..., C^{d-1}x\}, \{x, ..., C^{d-1}x\}) \in \mathbb{Z} \setminus \{0\}$, and moreover $G(\{(C-tI)^k x, ..., (C-tI)^k C^{d-1}x\}, \{x, ..., C^{d-1}x\}) \in \mathbb{Z}[t]$, where as usual $\mathbb{Z}[t]$ denotes the ring of all polynomials in t with integer coefficients. Hence $M\chi(t)^k \in \mathbb{Z}[t]$ for each $k \in \mathbb{N}$. We show that $\chi(t)$ has integer coefficients. Since $M\chi(t) \in \mathbb{Z}[t]$, $\chi(t)$ has rational coefficients, so, after some arithmetic, we may write $\chi(t) = q(t)/m$, where $q(t) \in \mathbb{Z}[t]$, $m \in \mathbb{N}$, and $q(t)$ and m are relatively prime in the ring $\mathbb{Z}[t]$. If $m = \pm 1$, $\chi(t) \in \mathbb{Z}[t]$, and otherwise we may factor M as $M = l^{k_0}M'$, where l is a prime divisor of m (in $\mathbb{Z}$), $m = lg$, and l does not divide M'. But then, if $k > k_0$ we have $M\chi(t)^k = Mq(t)^k/m^k = M'q(t)^k/(l^{k-k_0}g^k)$ and so l^{k-k_0} divides $M'q(t)^k$ in $\mathbb{Z}[t]$. Since $\mathbb{Z}[t]$ is a Euclidian ring [14, p. 70] and l is a prime in $\mathbb{Z}[t]$, this is a contradiction. Thus we conclude that $\chi(t) \in \mathbb{Z}[t]$.

To show $a_d = \pm 1$, we apply (23) for $L = (C|_{\mathcal{S}_x})^{-k}$, $k \in \mathbb{N}$, and the x_i and y_j as before, to obtain

$$\det(C|_{\mathcal{S}_x})^{-k} = \frac{G(\{C^{-k}x, ..., C^{d-k-1}x\}, \{x, ..., C^{d-1}x\})}{G(\{x, ..., C^{d-1}x\}, \{x, ..., C^{d-1}x\})}.$$

It follows that $M \det(C|_{\mathcal{S}_x})^{-k} \in \mathbb{Z}$ for each $k \in \mathbb{N}$. Using an argument similar to that above, we see that $a_d^{-1} = (-1)^d \det(C|_{\mathcal{S}_x})^{-1} \in \mathbb{Z}$. Therefore $a_d = \pm 1$. ∎

Definition 5.6. We say that a matrix $A \in M_n(\mathbb{R})$ has property $\mathcal{P}$ if for each $1 \leqslant k \leqslant n$, every product of k of its eigenvalues (with each eigenvalue repeated no more times than its corresponding algebraic multiplicity) is not a root of unity.

PROPOSITION 5.7. *Let* $A \in M_n'(\mathbb{R})$ *and let* $\{v_i\}$ *be a basis for* $\mathbb{R}^n$. *If* $\mathscr{U} = \mathscr{U}_{D_A, T_{v_i}}$ *and* A *has property* $\mathscr{P}$, *then*

(i) $T_x \mathscr{U} \subseteq \mathscr{U}$ *if and only if* $x = 0$, *and*

(ii) $B \in \mathscr{U}$ *has the property that* $B^p \in \mathscr{U}$ *for all* $p \in \mathbb{Z}$ *if and only if* B *is either an integral power of* D_A *or belongs to the group* $\mathscr{G}(\{T_{v_i}\})$ *generated by* $\{T_{v_1}, ..., T_{v_n}\}$.

Proof. The equivalence in (i) follows from Lemmas 5.4 (ii) and Proposition 5.5.

To establish (ii), assume $B^p \in \mathscr{U}$, $p \in \mathbb{Z}$. Then by Lemma 5.4 (iii), $B = D_A^q T_x$ for some $q \in \mathbb{Z}$, $x \in \mathbb{R}^n$ such that $A^{pq} x \in \operatorname{span}_{\mathbb{Z}}\{v_i\}$ for all $p \in \mathbb{Z}$. Suppose $x \neq 0$ and $q \neq 0$. Then by Proposition 5.5 with $C = A^q$, we obtain that the characteristic polynomial of $C|_{\mathscr{S}_x}$ has constant term equal to ± 1. Thus some product of eigenvalues of A^q is ± 1, and hence some product of eigenvalues of A is a root of unity. Hence A does not have property $\mathscr{P}$, which contradicts our hypothesis. So either $x = 0$ or $q = 0$. ∎

THEOREM 5.8. *Let* $A, B \in M_n'(\mathbb{R})$, *let* $\{v_i\}$ *and* $\{w_i\}$ *be bases for* $\mathbb{R}^n$, *and suppose that* A *has property* $\mathscr{P}$. *Then* $(D_A, T_{v_i}) \approx (D_B, T_{w_i})$ *if and only if there exists a unitary operator* $V \in B(L^2(\mathbb{R}^n))$ *such that* $V D_A V^* = D_{B^{\pm 1}}$ *and* $V \mathscr{G}(\{T_{v_i}\}) V^* = \mathscr{G}(\{T_{w_i}\})$.

Proof. If there exists a unitary V with the above properties, then it is clear that $\mathscr{U}_{D_A, T_{v_i}}$ and $\mathscr{U}_{D_B, T_{w_i}}$ are weakly equivalent.

Going the other way, we suppose that $\mathscr{U}_{D_A, T_{v_i}}$ and $\mathscr{U}_{D_B, T_{w_i}}$ are weakly equivalent via a unitary operator $V \in B(L^2(\mathbb{R}^n))$ and let $C = V D_A V^*$. By Proposition 5.7 (ii), either $C = D_B^q$ for some $q \in \mathbb{Z}$ or $C = T_x$ for some $x \in \operatorname{span}_{\mathbb{Z}}\{w_i\}$. Since $D_A \mathscr{U}_{D_A, T_{v_i}} = \mathscr{U}_{D_A, T_{v_i}}$ we have, upon conjugating this equation by V, $C \mathscr{U}_{D_B, T_{w_i}} = \mathscr{U}_{D_B, T_{w_i}}$. If $C = T_x$ then Proposition 5.7 (i) implies that $x = 0$, $C = I$, $D_A = I$, and $A = I$, a contradiction since A has property $\mathscr{P}$. Hence $C = D_B^q$. For $1 \leqslant i \leqslant n$, let $R_i = V T_{v_i} V^*$. Then $R_i^p \in \mathscr{U}_{D_B, T_{w_i}}$, $p \in \mathbb{Z}$, so Proposition 5.7 (ii) implies that either R_i is an integral power of D_B or $R_i \in \mathscr{G}(\{T_{w_i}\})$. But if R_i were an integral power of D_B, then we would have $R_i \mathscr{U}_{D_B, T_{w_i}} \subseteq \mathscr{U}_{D_B, T_{w_i}}$, and it would follow that $T_{v_i} \mathscr{U}_{D_A, T_{v_i}} \subseteq \mathscr{U}_{D_A, T_{v_i}}$. Then Proposition 5.7 (i) would imply $v_i = 0$, a contradiction. Hence $R_i \in \mathscr{G}(\{T_{w_i}\})$. Thus $V \mathscr{G}(\{D_A\}) V^* \subseteq \mathscr{G}(\{D_B\})$ and $V \mathscr{G}(\{T_{v_i}\}) V^* \subseteq \mathscr{G}(\{T_{w_i}\})$. Since $\mathscr{U}_{D_B, T_{w_i}} = \mathscr{G}(\{D_B\}) \mathscr{G}(\{T_{w_i}\})$, the two groups on the right have only I in common by Lemma 5.2 (ii), and the map $G \to V G V^*$ sends $\mathscr{U}_{D_A, T_{v_i}}$ onto $\mathscr{U}_{D_B, T_{w_i}}$, it follows that $V \mathscr{G}(\{D_A\}) V^* = \mathscr{G}(\{D_B\})$ and $V \mathscr{G}(\{T_{v_i}\}) V^* = \mathscr{G}(\{T_{w_i})\}$. In particular, C is a generator of

$\mathcal{G}(\{D_B\})$, so $D_B = C^{\ell}$ for some $\ell \in \mathbb{Z}$. Thus $D_B^{\ell q} = D_B$ and so $D_{B^{\ell q - 1}} = I = T_0$. By Lemma 5.2 (ii), $B^{\ell q - 1} = I$ and since $D_A = V^* D_B V$, we conclude easily that $A^{\ell q - 1} = I$. Since A has property $\mathcal{P}$, we have $\ell q = 1$, and since $\ell, q \in \mathbb{Z}$ this means $q = \pm 1$ and $C = D_{B^{\pm 1}}$. ∎

COROLLARY 5.9. *Suppose $A, B \in M_n'(\mathbb{R})$ are expansive matrices and let $\{v_i\}$ and $\{w_i\}$ be bases for $\mathbb{R}^n$. Then $(D_A, T_{v_i}) \approx (D_B, T_{w_i})$ if and only if there exist a unitary operator $V \in B(L^2(\mathbb{R}^n))$ and a basis $\{w_i'\}$ for $\mathbb{R}^n$ such that $\text{span}_{\mathbb{Z}}\{w_i\} = \text{span}_{\mathbb{Z}}\{w_i'\}$, $VD_A V^* = D_B$, and $VT_{v_i}V^* = T_{w_i'}$, $i = 1, ..., n$. (In other words, $(D_A, T_{v_i}) \approx (D_B, T_{w_i})$ if and only if there exist a basis $\{w_i'\}$ for $\mathbb{R}^n$ with $\text{span}_{\mathbb{Z}}\{w_i\} = \text{span}_{\mathbb{Z}}\{w_i'\}$ and such that $(D_A, T_{v_i}) \sim (D_B, T_{w_i})$).*

Proof. If there exists a unitary V with the above properties, then it is clear that $(D_A, T_{v_i}) \approx (D_B, T_{w_i})$.

To establish the other implication let us assume that $A, B \in M_n'(\mathbb{R})$ are expansive matrices and $(D_A, T_{v_i}) \approx (D_B, T_{w_i})$. Since every expansive matrix has property $\mathcal{P}$, it follows by Theorem 5.8 that $VD_A V^* = D_F$ where either $F = B$ or $F = B^{-1}$. Also by Theorem 5.8, for $i = 1, ..., n$, there exists $w_i \in \text{span}_{\mathbb{Z}}\{w_i\}$ such that $VT_{v_i}V^* = T_{w_i'}$ and $\mathcal{G}(\{T_{w_i'}\}) = \mathcal{G}(\{T_{w_i}\})$. This implies by Lemma 5.3 that $\text{span}_{\mathbb{Z}}\{w_i'\} = \text{span}_{\mathbb{Z}}\{w_i\}$ and hence that $\{w_i'\}$ is a basis for $\mathbb{R}^n$. Thus $(D_A, T_{v_i}) \sim (D_F, T_{w_i'})$. Hence by Theorem 3.1, F is similar to A, so F is also expansive. Since B^{-1} is not expansive, we must have $F = B$, and so $VD_A V^* = D_B$. ∎

The following propositions add some additional perspective to this theory.

PROPOSITION 5.10. *Let $A \in M_n'(\mathbb{R})$ and let $\{v_i\}$ be a basis of $\mathbb{R}^n$. Then there exists $B \in M_n'(\mathbb{R})$ such that $(D_A, T_{v_i}) \sim (D_B, T_{e_i})$, where $\{e_i\}$ is the canonical basis for $\mathbb{R}^n$.*

Proof. Define $S \in M_n(\mathbb{R})$ by $Se_i = v_i$, $i = 1, ..., n$, and set $B = S^{-1}AS$ and $V = D_{S^{-1}}$. Then $V^* D_A V = D_B$ and $V^* T_{v_i} V = T_{e_i}$, $i = 1, ..., n$. ∎

PROPOSITION 5.11. *Let A, and B be expansive matrices in $M_n'(\mathbb{R})$. Then $(D_A, T_{e_i}) \approx (D_B, T_{e_i})$ if and only if there is a matrix $C \in M_n'(\mathbb{R})$ with the property that both C and C^{-1} have integer entries and $B = C^{-1}AC$.*

Proof. Apply Corollary 5.9, obtaining a basis $\{w_i'\}$ for $\mathbb{R}^n$ with $\text{span}_{\mathbb{Z}}\{w_i'\} = \text{span}_{\mathbb{Z}}\{e_i\}$ such that $(D_A, T_{e_i}) \sim (D_B, T_{w_i'})$. By Theorem 3.1, $B = C^{-1}AC$ where C is the matrix such that $Cw_i' = e_i$, $i = 1, ..., n$. Since $\text{span}_{\mathbb{Z}}\{w_i'\} = \text{span}_{\mathbb{Z}}\{e_i\}$, C must have the required form. ∎

The above results give rise to the following interesting question.

Problem 5.12. If $A, B \in M'_n(\mathbb{R})$ are matrices which do not have property $\mathscr{P}$, how can one characterize the equivalence $(D_A, T_{e_i}) \approx (D_B, T_{e_i})$?

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