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ON THE INNER AUTOMORPHISMS OF FINITE TRANSFORMATION SEMIGROUPS

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If the group of inner automorphisms of a semigroup S of transformations of a finite n-element set contains an isomorphic copy of the alternating group Alt_n, then S is an S_n-normal semigroup and all the automorphisms of S are inner.

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1. Introduction

Given a semigroup S of transformations of a set $X_n=\{1,2,..,n\}$, denote by G_S the subgroup of the symmetric group S_n of all the permutations h of X_n satisfying $hSh^{-1}\subseteq S$. Therefore for each $h\in G_S$, the mapping $\phi_h: S\to S$ defined by $\phi_h(\alpha)=h\alpha h^{-1}$, for $\alpha\in S$, is an automorphism of S. Such an automorphism of S is termed *inner* [5] and the set of all inner automorphisms of S, $\mathrm{Inn}\,S=\{\phi_h: h\in G_S\}$, forms a subgroup of the group $\mathrm{Aut}\,S$ of all automorphisms of S.

Observe that if $S=T_n$, the semigroup of all total transformations of X, then $G_S=S_n$. A subsemigroup S of T_n is said to be *S_n-normal* if $G_S=S_n$. In this case all the automorphisms of S are inner, and $\mathrm{Aut}\,S=\mathrm{Inn}\,S\cong S_n$ [6].

The main result of this paper asserts that if G_S contains the alternating group Alt_n then $G_S=S_n$, so that S is an S_n-normal semigroup, and $\mathrm{Aut}\,S=\mathrm{Inn}\,S\approx S_n$. Therefore, there is no $S\subseteq T_n$ such that $G_S=\mathrm{Alt}_n$.

We generally use letters h, p, g to denote permutations of X_n, and α, β, γ, δ to denote non-permutations in T_n. In the following series of results we prove the theorem stated below.

Theorem. *Let S be a subsemigroup of T_n, $n\geq 3$. If the group* $\mathrm{Inn}\,S$ *contains a subgroup G isomorphic to* Alt_n *then* $\mathrm{Aut}\,S=\mathrm{Inn}\,S\cong S_n$, *and S is an S_n-normal semigroup.*

Given $\alpha\in T_n$ and a subgroup G of S_n, let $\langle\alpha: G\rangle=\langle\{h\alpha h^{-1}: h\in G\}\rangle$ be the subsemigroup of T_n generated by all the conjugates of α by the elements of G. Observe that if $\beta\in\langle\alpha: G\rangle$, then $\beta=h_1\alpha h_1^{-1}h_2\alpha h_2^{-1}\dots h_k\alpha h_k^{-1}$ for some $h_1,h_2,\dots,h_k\in G$, and so for any $h\in G$, $h\beta h^{-1}=hh_1\alpha h_1^{-1}h^{-1}hh_2\alpha h_2^{-1}h^{-1}\dots hh_k\alpha h_k h^{-1}=(hh_1)\alpha(hh_1)^{-1}(hh_2)\alpha(hh_2)^{-1}\dots(hh_k)\alpha(hh_k)^{-1}\in\langle\alpha: G\rangle$. Therefore $\langle\beta: G\rangle\subseteq\langle\alpha: G\rangle$.

Lemma 1. *Let $G_1 \leqq G_2 \leqq S_n$ and $[G_2: G_1] = 2$. Let $\alpha \in T_n - S_n$. Then $\langle \alpha: G_1 \rangle = \langle \alpha: G_2 \rangle$ if and only if there is an $h \in G_2 - G_1$ such that $h\alpha h^{-1} \in \langle \alpha: G_1 \rangle$.*

Proof. If $\langle \alpha: G_1 \rangle = \langle \alpha: G_2 \rangle$ then for any $h \in G_2$, $h\alpha h^{-1} \in \langle \alpha: G_1 \rangle$. To show the converse assume that $h \in G_2 - G_1$ is such that $\beta = h\alpha h^{-1} \in \langle \alpha: G_1 \rangle$. Let $p \in G_2 - G_1$. It suffices to show that $p\alpha p^{-1} \in \langle \alpha: G_1 \rangle$. Since h, $p \in G_2 - G_1$ and $[G_2: G_1] = 2$, we have $G_1 h = G_1 p$, so there exists $q \in G_1$, with $q = ph^{-1}$. Therefore $p\alpha p^{-1} = qh\alpha(qh)^{-1} = qh\alpha h^{-1} q^{-1} = q\beta q^{-1} \in \langle \beta: G_1 \rangle \subseteq \langle \alpha: G_1 \rangle$, as required. □

The following is used to show that if $\alpha \in T_n - S_n$ then $\langle \alpha: \mathrm{Alt}_n \rangle = \langle \alpha: S_n \rangle$.

Corollary 2. *$\langle \alpha: \mathrm{Alt}_n \rangle = \langle \alpha: S_n \rangle$ if and only if there exists an odd permutation h of X_n such that $h\alpha h^{-1} \in \langle \alpha: \mathrm{Alt}_n \rangle$.*

Recall that a subgroup G of S_n is said to be *k-transitive* if for any two k-subsets A and B of X_n and any bijection t from A onto B, there exists $h \in G$ such that $h(a) = t(a)$ for every $a \in A$. We say that a subgroup G of S_n is *k-block-transitive* if for any two k-subsets A and B of X_n there exists $h \in G$ such that $h(A) = B$. Thus any k-transitive semigroup is at least k-block-transitive. For example, Alt_n is $(n-2)$-transitive [4, 10.4.6], and for all $1 \leqq k \leqq n-1$, Alt_n is k-block transitive.

Given a transformation α of X_n denote by $\pi(\alpha)$ the partition of X_n determined by α such that a and b are in the same class of $\pi(\alpha)$ if and only if $\alpha(a) = \alpha(b)$. Let $\mathrm{im}\,\alpha = \alpha(X_n)$ be the image of α. Note that if $h \in S_n$ then $\pi(h\alpha h^{-1}) = h(\pi(\alpha)) = \{h(A): A \in \pi(\alpha)\}$, and $\mathrm{im}(h\alpha h^{-1}) = h(\mathrm{im}\,\alpha)$.

Lemma 3. *Let $G \leqq S_n$ be a k-block transitive group. Then for any $\alpha \in T_n - S_n$ with $|\mathrm{im}\,\alpha| = k$, $\langle \alpha: G \rangle$ contains an idempotent β with $\pi(\beta) = \pi(\alpha)$.*

Proof. Let $\alpha_1 (=\alpha), \alpha_2, \alpha_3, \ldots$ be conjugates of α by elements of G such that $\mathrm{im}\,\alpha_i$ is a transversal of $\pi(\alpha_{i+1})$ (k-block transitivity of G insures their existence). Consider all the products of the form $\alpha_1, \alpha_2\alpha_1, \alpha_3\alpha_2\alpha_1, \ldots$ Since $\langle \alpha: G \rangle$ is finite there exist integers $m < j$ such that $\alpha_j\alpha_{j-1} \ldots \alpha_{m+1}\alpha_m \ldots \alpha_1 = \alpha_m \ldots \alpha_1$. Let $\delta = \alpha_j \ldots \alpha_{m+1}$ and $\gamma = \alpha_m \ldots \alpha_1$. Then $\delta\gamma = \gamma$ so $\mathrm{im}\,\delta \supseteq \mathrm{im}\,\gamma$, and since $|\mathrm{im}\,\delta| = |\mathrm{im}\,\alpha| = |\mathrm{im}\,\gamma|$ we have that $\mathrm{im}\,\delta = \mathrm{im}\,\gamma$. Thus δ is the identity on its image, and so δ is an idempotent having $\mathrm{im}\,\delta = \mathrm{im}\,\alpha_j$ and $\pi(\delta) = \pi(\alpha_{m+1})$. Let $h \in G$ be such that $\alpha_{m+1} = h\alpha h^{-1}$, then $\beta = h^{-1}\delta h$ is the required idempotent. Indeed $\beta^2 = h^{-1}\delta h h^{-1}\delta h = h^{-1}\delta^2 h = h^{-1}\delta h = \beta$ and $\pi(\beta) = \pi(h^{-1}\delta h) = h^{-1}(\pi(\delta)) = h^{-1}(\pi(\alpha_{m+1})) = h^{-1}(\pi(h\alpha h^{-1}) = h^{-1}(\pi(h(\alpha))) = \pi(\alpha)$. □

Since Alt_n is k-block-transitive for any $1 \leqq k \leqq n-1$ we have the following.

Corollary 4. *$\langle \alpha: \mathrm{Alt}_n \rangle$ contains an idempotent β with $\pi(\beta) = \pi(\alpha)$.*

We say that α has a *partition of type* $1^{k_1}2^{k_2}\ldots r^{k_r}$ if $\pi(\alpha)$ has k_i classes of size i,

$i=1,\dots,r$[1]. Note that $\sum_{i=1}^{r} ik_i=n$ and we do not exclude the possibility that $k_i=0$ for some i.

Lemma 5. *Let $\alpha\in T_n-S_n$ be an idempotent. There exists an $h\in S_n-\mathrm{Alt}_n$ such that $h\alpha h^{-1}\in\langle\alpha:\mathrm{Alt}_n\rangle$, $n\geqq 3$.*

Proof. Assume that there exist $x,y\in \operatorname{im}\alpha$ such that $\alpha^{-1}(x)=\{x\}$ and $\alpha^{-1}(y)=\{y\}$. Then for the transposition $h=(x,y)$ we have $h\alpha h^{-1}=\alpha\in\langle\alpha:\mathrm{Alt}_n\rangle$. Now suppose $\pi(\alpha)$ contains a class A having $|A|\geqq 3$. Let $a,b\in A-\operatorname{im}\alpha$. Then for $h=(a,b)$ we have $h\alpha h^{-1}=\alpha\in\langle\alpha:\mathrm{Alt}_n\rangle$.

If none of the above holds then α has a partition of type $1^0 2^k=2^k$ ($k=n/2$, n is even) or $1^1 2^k$ ($k=(n-1)/2$, n is odd). Let α_1,α_2 be idempotents in T_{2k} and T_{2k+1} respectively, $\alpha_1=[1,1,3,3,\dots,2k-1,2k-1]$ and $\alpha_2=[1,1,3,3,\dots,2k-1,2k-1,2k+1]$ (we write $[a_1,a_2,\dots,a_l]$ for a transformation mapping i to a_i. We may assume without loss of generality that α equals to either α_1 or α_2. It is easy to verify that for $h=(12)$ and $n\geqq 5$ we have

$$h\alpha_i h^{-1}=(12)(35)\alpha_i(35)(12)\alpha_i\in\langle\alpha:\mathrm{Alt}_n\rangle.$$

If $n=4$, then $\alpha_1=[1,1,3,3]$, and for $h=(12)$,

$$h\alpha_i h^{-1}=((132)\alpha_1(123))((134)\alpha_1(143))\alpha_1\in\langle\alpha_1:\mathrm{Alt}_4\rangle.$$

If $n=3$, $\alpha_2=[1,1,3]$, and for $h=(12)$,

$$h\alpha_2 h^{-1}=((132)\alpha_2(123))((123)\alpha_2(132)\alpha_2)^2\in\langle\alpha_2:\mathrm{Alt}_n\rangle. \qquad \square$$

Proposition 6. *Let $\alpha\in T_n$, $n\geqq 3$. Then $\langle\alpha:S_n\rangle$.*

Proof. Observe that we only need to show that $\langle\alpha:S_n\rangle\subseteq\langle\alpha:\mathrm{Alt}_n\rangle$. If $\alpha\in\mathrm{Alt}_n$ then $\langle\alpha:S_n\rangle\leqq\mathrm{Alt}_n\trianglelefteq S_n$. Also $\langle\alpha:\mathrm{Alt}_n\rangle\trianglelefteq\mathrm{Alt}_n$, and since Alt_n is simple for $n\neq 4$ [4, 10.8.7] we have that $\langle\alpha:\mathrm{Alt}_n\rangle=\mathrm{Alt}_n$ if $\alpha\neq(1)$ and $\langle(1):\mathrm{Alt}_n\rangle=\{(1)\}$ (provided $n\neq 4$). If $n=4$, $\alpha\neq(1)$ and $\langle\alpha:\mathrm{Alt}_4\rangle\neq\mathrm{Alt}_4$ then $\langle\alpha:\mathrm{Alt}_4\rangle=\mathbf{V}$, the 4-*group*, and $\alpha\in\mathbf{V}$. Since $\mathbf{V}\trianglelefteq S_4$, $\langle\alpha:S_4\rangle\leqq\mathbf{V}$ also, and therefore $\langle\alpha:S_4\rangle\subseteq\langle\alpha:\mathrm{Alt}_4\rangle$, as required.

If α is an odd permutation then for any $q\in S_n-\mathrm{Alt}_n$, $q=\alpha(\alpha^{-1}q)$, $\alpha^{-1}q\in\mathrm{Alt}_n$, and $q\alpha q^{-1}=\alpha(\alpha^{-1}q)\alpha(\alpha^{-1}q)^{-1}\alpha^{-1}\in\langle\alpha:\mathrm{Alt}_n\rangle$, so $\langle\alpha:S_n\rangle\subseteq\langle\alpha:\mathrm{Alt}_n\rangle$ again.

Now let $\alpha\in T_n-S_n$. By Corollary 4, $\langle\alpha:\mathrm{Alt}_n\rangle$ contains an idempotent β with $\pi(\beta)=\pi(\alpha)$. By Lemma 5 and Corollary 2, $\langle\alpha:\mathrm{Alt}_n\rangle\supseteq\langle\beta:\mathrm{Alt}_n\rangle=\langle\beta:S_n\rangle=\langle\alpha:S_n\rangle$ (for a transformation γ the semigroup $\langle\gamma:S_n\rangle$ comprises all $\delta\in T_n$ having $\pi(\delta)\supseteq Q$, a partition of the same type as $\pi(\gamma)$ [2]). $\square$

Corollary 7. *There is no $S\subseteq T_n$ such that $G_S=\mathrm{Alt}_n$.*

Proof. Suppose $\mathrm{Alt}_n\subseteq G_S$. Then by Proposition 6, for any $\alpha\in S$, $h\in S_n$, we have that $h\alpha h^{-1}\in\langle\alpha:S_n\rangle=\langle\alpha:\mathrm{Alt}_n\rangle\subseteq S$, that is $h\in G_S$ and $G_S=S_n$. $\square$

Now to prove our main Theorem suppose that $G \leq \operatorname{Inn} S$ such that $G \cong \mathrm{Alt}_n$, Let $\bar{G} = \{h \in S_n : \phi_h \in G\}$. Then $\bar{G} \leqq G_S \leqq S_n$, and the order of $\bar{G}$ is at least that of Alt_n. Therefore $\bar{G}$ contains Alt_n, and by Corollary 7, $G_S = S_n$.

We note that the above result is not necessarily true for semigroups of transformations of infinite sets. For example, let $X = \mathbf{Z}$ be the set of all integers and $\alpha: \mathbf{Z} \to \mathbf{Z}$ such that $\alpha(a) = 2a$, for all $a \in \mathbf{Z}$. Let $S_{\mathbf{Z}}$ be the symmetric group on $\mathbf{Z}$. The alternating subgroup $\mathrm{Alt}_{\mathbf{Z}}$ of $S_{\mathbf{Z}}$ consists of all the finite even permutations of $\mathbf{Z}$. Then $\langle \alpha : S_{\mathbf{Z}} \rangle = \{\beta : \mathbf{Z} \to \mathbf{Z} \mid \beta \text{ is } 1-1 \text{ and } |\mathbf{Z} - \operatorname{im} \beta| = \aleph_0\}$ [3]. In particular $\langle \alpha : S_{\mathbf{Z}} \rangle$ contains β defined by $\beta(a) = 2a - 1$ for all $a \in \mathbf{Z}$. Observe that for all $a \in \mathbf{Z}$, $\alpha(a) \neq \beta(a)$. Since any $h \in \mathrm{Alt}_{\mathbf{Z}}$ moves at most a finite number of points, $\beta \notin \langle \alpha : \mathrm{Alt}_{\mathbf{Z}} \rangle$.

For a transformation α of X let shift $\alpha = |\{x \in X : \alpha(x) \neq x\}|$. Let v be an infinite cardinal not exceeding $|X|^+$, the cardinal successor of $|X|$, and let $\mathrm{Sym}(X, v)$ be the subgroup of all permutations in S_X whose shift is less than v.

Conjecture 1. If shift $\alpha = u$ then $\langle \alpha : \mathrm{Sym}(X, w) \rangle = \langle \alpha : S_X \rangle$ for all $w \geqq u^+$.
2. There is no semigroup S of transformations of X having $G_S = \mathrm{Sym}(X, |X|)$.

Observe that permutations h and p in G_S give rise to equal automorphisms ϕ_h and ϕ_p if and only if $h^{-1}p$ is in the centralizer $C(S)$ of S, $C(S) = \{\alpha \in T_n : \alpha\beta = \beta\alpha \text{ for all } \beta \in S\}$. Thus G_S is isomorphic to the group $\operatorname{Inn} S$ of the inner automorphisms of S if and only if $C(S) \cap G_S$ consists of the identity permutation. The results of this paper in conjunction with the above observations give rise to the following.

Problem 1. Characterize these subgroups G of S_n having $G = G_S$ for some subsemigroup S of S_n.
2. Given that $G = G_T$ for some $T \subseteq T_n$ characterize all $S \subseteq T_n$ such that $G_S = G$.
3. Characterize these subsemigroups S of T_n having $|C(S) \cap G_S| = 1$.

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