ON POWER BOUNDED OPERATORS

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Abstract. In this paper we generalize the following consequence of a well-known result of Nagy: if \( T \) and \( T^{-1} \) are power bounded operators, then \( T \) is a polynomially bounded operator.

Let \( \mathcal{H} \) be a separable, infinite dimensional, complex Hilbert space, and let \( \mathcal{L}(\mathcal{H}) \) denote the algebra of all bounded linear operators on \( \mathcal{H} \). Recall that an operator \( T \in \mathcal{L}(\mathcal{H}) \) is called \textit{power bounded} (notation: \( T \in \mathcal{P}W(\mathcal{H}) \)) if there exists a constant \( M (\geq 1) \) such that

\[
\| T^n \| \leq M, \quad n \in \mathbb{N},
\]

and \( T \) is called \textit{polynomially bounded} (notation: \( T \in \mathcal{P}B(\mathcal{H}) \)) if there exists a constant \( M (\geq 1) \) such that

\[
\| p(T) \| \leq M \| p \|_{\infty}
\]

for every polynomial \( p \), where \( \| p \|_{\infty} = \sup \{ |p(z)| : z \in \mathbb{C}, \ |z| \leq 1 \} \). The smallest number \( M \) satisfying (1) (resp., (2)) is called the \textit{power bound} (resp., the \textit{polynomial bound}) of \( T \) and will be denoted by \( M_w(T) \) (resp., \( M_p(T) \)), or simply \( M_w \) (resp., \( M_p \)) when no confusion is possible. One knows (cf. [1, 2, 4]) that \( \mathcal{P}W(\mathcal{H}) \) strictly contains the class \( \mathcal{P}B(\mathcal{H}) \), but there is a theorem of Nagy [3] which says that every \( T \in \mathcal{P}W(\mathcal{H}) \) such that \( T^{-1} \) exists and belongs to \( \mathcal{P}W(\mathcal{H}) \) is similar to a unitary operator, and therefore is polynomially bounded. The purpose of this note is to establish the following two stronger results than the above-mentioned consequence of Nagy’s theorem.

Theorem 1.1. Suppose \( T \in \mathcal{P}W(\mathcal{H}) \) (with \( M_w(T) > 1 \)) and the following inequality holds for some positive number \( \alpha \) and a strictly increasing sequence \( \{ n_k \} \subset \mathbb{N} : \)

\[
1/n_k \sum_{j=0}^{n_k} T^{*j}T^j \geq \alpha (I - P_{\text{ker}(T)}),
\]

where \( P_{\text{ker}(T)} \) is the (orthogonal) projection on the kernel of \( T \). Then \( T \in \mathcal{P}B(\mathcal{H}) \) and the polynomial bound \( M_p \) of \( T \) satisfies

\[
M_p \leq M_w(T)^3 \left( \frac{M_w(T)^2 - 1}{\alpha n M_w(T)} \right)^{1/2} + 1.
\]

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Theorem 1.2. Suppose $T \in \mathcal{P}W(\mathcal{H})$ and the following inequality holds for some positive number $\alpha$ and a strictly increasing sequence $\{t_k\}_{k \in \mathbb{N}}$ of real numbers converging to 1:

\[(1 - t_k) \sum_{j=0}^{\infty} t_k^j T^* T^j \geq \alpha (I - P_{\ker(T)}).\]  

Then $T \in \mathcal{P}B(\mathcal{H})$, and the polynomial bound $M_p$ of $T$ satisfies

\[M_p \leq \left( \frac{14}{\alpha} \right)^{1/2} M_w^3.\]  

As mentioned above, the following is an immediate consequence of either Theorem 1.1 or Theorem 1.2.

Corollary 1.3 (Nagy [4]). If $T \in \mathcal{P}W(\mathcal{H})$ is invertible and $T^{-1} \in \mathcal{P}W(\mathcal{H})$, then $T$ is polynomially bounded.

In order to prove Theorem 1.1, we use the following lemma, which is well known [5].

Lemma 1.4. Suppose $S \in \mathcal{L}(\mathcal{H})$ is such that $S^m$ is a contraction for some integer $m \geq 2$. Then $S$ is similar to a contraction, and, in particular,

\[A = (I + S^* S + ... + S^{*(m-1)} S^{(m-1)})^{1/2}\]

is an invertible operator that satisfies

\[\|ASA^{-1}\| \leq 1.\]

Proof. Clearly $A$ is an invertible selfadjoint operator. To establish (8) it is enough to check that

\[\|ASA^{-1}h\| \leq \|h\|, \ h \in \mathcal{H}.\]

For a given $h$, define $g = A^{-1}h$, and hence (9) becomes equivalent to

\[\langle A^2 Sg, Sg \rangle \leq \langle A^2 g, g \rangle.\]

Using (7), we see that (10) is equivalent to

\[\sum_{j=1}^{m} \|S^j g\|^2 \leq \sum_{j=0}^{m-1} \|S^j g\|^2,\]

which is true since $\|S^m g\| \leq \|g\|$. \hfill \qed

Proof of Theorem 1.1. For brevity we write $M = M_w(T) > 1$. For each $n \in \mathbb{N}$, set $\alpha_n = M^{1/n}$, $\beta_n = \alpha_n^{-1}$, and note that $\beta_n < 1 < \alpha_n$. Since $\|(\beta_n T)^n\| \leq 1$ for each $n \in \mathbb{N}$, we may apply Lemma 1.4 to obtain for each $n \in \mathbb{N}$ a contraction $C_n$ such that

\[\beta_n T = A_n^{-1} C_n A_n,\]

where $A_n = (\sum_{j=0}^{n-1} \beta_n^j T^{*j} T^j)^{1/2}$. Consider now an arbitrary polynomial $p(z) = a_0 + a_1 z + a_2 z^2 + ... + a_l z^l$. Then

\[p(T) = p(\alpha_n A_n^{-1} C_n A_n) = A_n^{-1} p(\alpha_n C_n) A_n.\]
Applying the von Neumann inequality to \( C_n \) and the polynomial \( q_n(z) = p(\alpha_n z) \), we conclude from (12) that

\[
\|p(T)\| \leq \|A_n^{-1}\| \|A_n\||q_n||_\infty, \quad n \in \mathbb{N}.
\]

Let us observe now that for each \( n \in \mathbb{N} \),

\[
\|A_n\|^2 = \|A_n^2\| \leq \sum_{j=0}^{n-1} \beta_{n_k}^{2j} M^2 = M^2(1 - \beta_{n_k}^{2n})/(1 - \beta_{n_k}^2) = M^2(1 - M^{-2})/(1 - \beta_{n_k}^2),
\]

so

\[
\|A_n\| \leq (M^2 - 1)^{1/2}/(1 - \beta_{n_k}^2)^{1/2}.
\]

Moreover, for each \( n \in \mathbb{N} \), \( \|A_n^{-1}\| = \gamma(A_n)^{-1} \), where \( \gamma(A_n) \) is the greatest number \( \gamma > 0 \) with the property that \( \|A_n h\| \geq \gamma \|h\| \) for all \( h \in \mathcal{H} \). Equivalently,

\[
\langle A_n^2 h, h \rangle \geq \gamma(A_n)^2 \langle h, h \rangle, \quad h \in \mathcal{H}, \quad n \in \mathbb{N}.
\]

Consider now the case that \( ker(T) = \{0\} \). Let \( \{n_k\} \) be the sequence from (3). Then

\[
A_{n_k}^2 = \sum_{j=0}^{n_k-1} \beta_{n_k}^{2j} T^* T^j \geq \beta_{n_k}^{2n_k} \sum_{j=0}^{n_k-1} T^* T^j \geq \beta_{n_k}^{2n_k} n_k \alpha I = n_k M^{-2} \alpha I, \quad k \in \mathbb{N}.
\]

Therefore, \( \gamma(A_{n_k}) \geq n_k^{1/2} M^{-1} \alpha^{1/2} \) for each \( k \in \mathbb{N} \), which implies that

\[
\|A_n^{-1}\| = \gamma(A_{n_k})^{-1} \leq n_k^{-1/2} M \alpha^{-1/2}.
\]

Thus, from (14) and (15) we get

\[
\|A_n\||A_n^{-1}\| \leq M(2M - 1)^{1/2} \alpha^{-1/2}/(n_k(1 - \beta_{n_k}^2))^{1/2}, \quad k \in \mathbb{N}.
\]

A simple continuity argument shows that for \( p \) fixed we have

\[
\lim_{k \to \infty} \|q_{n_k}\| = \|p\|_\infty.
\]

Going back to (13), and taking into account (16) and (17), we can let \( k \) go to infinity and obtain the inequality

\[
\|p(T)\| \leq M((M^2 - 1)/2lnM)^{1/2} \alpha^{-1/2}\|p\|_\infty
\]

by using the formula (from elementary calculus)

\[
\lim_{n \to \infty} (M^{2/n} - 1)n = 2ln M.
\]

Thus, in this case, \( T \in \mathcal{PB}(\mathcal{H}) \) and (4) is valid.

Let us consider now the general case. With respect to the decomposition \( \mathcal{H} = (ker T) \oplus (ker T)^\perp \), \( T \) has an operator matrix

\[
T = \begin{bmatrix} 0 & S \\ 0 & Q \end{bmatrix}
\]

where \( S : (ker T)^\perp \to (ker T)^\perp \) is a bounded linear operator, \( Q \in \mathcal{PW}((ker T)^\perp) \), and \( M_w(Q) \leq M_w(T) \). For each polynomial \( p \) one sees easily that

\[
p(T) = \begin{bmatrix} p(0)I & Sq(Q) \\ 0 & p(Q) \end{bmatrix}
\]
where \( q(z) = (p(z) - p(0))/z \). Therefore, since \( \|q\|_\infty \leq 2\|p\|_\infty \), it is sufficient to show that \( Q \) is polynomially bounded and has an appropriate polynomial bound. We want to use the first case, so let us observe that

\[
T^*kT^k \leq \begin{bmatrix} 0 & 0 \\ 0 & (\|S\|^2 + \|Q\|^2)Q^kQ^k-1 \end{bmatrix}, \quad k \in \mathbb{N}.
\]

But (3) and (20) together yield

\[
(\|S\|^2 + \|Q\|^2)/(n_k - 1) \sum_{j=0}^{n_k-1} Q^jQ^j \geq (\alpha - (\alpha + 1)/(n_k - 1))I_{(\ker T)^\perp}.
\]

In particular this says that if \( h \in \ker(Q) \cap (\ker T)^\perp \), then

\[
(\|S\|^2 + \|Q\|^2)/(n_k - 1)\langle h, h \rangle \geq (\alpha - (\alpha + 1)/(n_k - 1))\langle h, h \rangle,
\]

and letting \( k \) go to infinity we obtain that \( h = 0 \). Hence, \( Q \) satisfies the condition (3) in the case when \( \ker(Q) = \{0\} \) for \( \alpha' = (\alpha - \epsilon)/(\|S\|^2 + \|Q\|^2) > 0 \) and a subsequence \( \{n_k - 1\} \) for \( k \) large enough (depending upon \( \epsilon \)). Therefore, we obtain from the previous case,

\[
\|p(Q)\| \leq M((M^2 - 1)/2lnM)^{1/2}\alpha^{-1/2}(\|S\|^2 + \|Q\|^2)^{1/2}\|p\|_\infty,
\]

since \( \epsilon > 0 \) was arbitrary. Finally we get

\[
\|p(T)\| \leq (M((M^2 - 1)/2lnM)^{1/2}\alpha^{-1/2}(\|S\|^2 + \|Q\|^2)^{1/2}\|S\| + 1)\|p\|_\infty
\leq (M^2((M^2 - 1)/lnM)^{1/2}\alpha^{-1/2} + 1)\|p\|_\infty,
\]

which is what we wanted to show.

We want to consider now the continuous analog of Theorem 1.1.

**Proof of Theorem 1.2.** Let us define for \( T \in \mathcal{PW}(\mathcal{H}) \) and every \( t \in [0, 1) \) the self-adjoint invertible operator

\[
A_t = (1 - t)^{1/2}(\sum_{j=0}^{\infty} t^jT^*T^j)^{1/2}.
\]

First, observe that this operator is well-defined for \( T \in \mathcal{PW}(\mathcal{H}) \), and moreover

\[
\|A_t\|^2 = \|A_t^2\| \leq (1 - t)\sum_{j=0}^{\infty} t^j\|T^*T^j\|\leq M_w(T)^2.
\]

As before, let us consider the case when \( \ker(T) = \{0\} \). If (5) is satisfied, then \( \|A_t^{-1}\| \leq \alpha^{-1/2} \) at least for \( t = t_k \).

Now observe that for any \( h \in \mathcal{H} \) we have

\[
(1 - t)\|A_t^{-1}h\|^2 + t\|A_tT^*A_t^{-1}h\|^2 = \|h\|^2,
\]

which, in particular, says that \( t^{1/2}A_tT^*A_t^{-1} \) is a contraction. Hence we can use the idea from the proof of Theorem 1.1 to get that

\[
\|p(T)\| \leq \|A_{t_k}\|\|A_{t_k}^{-1}\|\|q_k\|_\infty, \quad k \in \mathbb{N},
\]

where \( q_k(z) = p(t_k^{-1/2}z) \) for any given polynomial \( p \). Letting \( k \) go to infinity we get the inequality

\[
\|p(T)\| \leq M_w(T)\alpha^{-1/2}\|p\|_\infty,
\]
which is what we wanted to show in the case \( \ker(T) = 0 \). In the general case, if \( T \) has the decomposition (19), by using the inequality (20) and the hypothesis (5), we have that

\[
\left( \|S\|^2 + \|Q\|^2 \right) (1 - t_k) \sum_{j=1}^{\infty} t_j^k Q^{j-1} Q^{j-1} \geq (\alpha - 1 + t_k) I_{(\ker(T))^\perp},
\]

which says, first, that \( \ker(Q) = 0 \) and thus that \( Q \) is as in the first case. Therefore, we finally get

\[
\|p(T)\| \leq M_w(T) \left\{ (3 + 4\|S\|^2) \left( \frac{\|S\|^2 + \|Q\|^2}{\alpha} \right) \right\}^{1/2} \|p\|_\infty
\]

\[
\leq \left( \frac{14}{\alpha} \right)^{1/2} M_w(T)^3 \|p\|_\infty,
\]

which was to be proved.

An easy corollary of Theorem 1.2 is the following generalization.

**Corollary 1.5.** Suppose \( T \in \mathcal{PW}(H) \) and the following inequality holds for some \( n \in \mathbb{N} \), some positive number \( \alpha \), and a strictly increasing sequence \( \{t_k\}_{k \in \mathbb{N}} \) of real numbers converging to 1:

\[
(1 - t_k) \sum_{j=0}^{\infty} t_j^k T^j T^j \geq \alpha (I - P_{\ker(T^n)}).
\]

(23)

Then \( T \) is polynomially bounded.

**Proof.** With respect to the decomposition \( H = \ker(T^n) \oplus (\ker(T^n))^\perp \), \( T \) has the operator matrix

\[
T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.
\]

Since \( \ker(T^n) \) is an invariant subspace for \( T \), the operator \( C \) must be zero. In addition, we have that

\[
T^n = \begin{bmatrix} 0 & E \\ 0 & F \end{bmatrix},
\]

where

\[
A^n = 0, \ F = D^n, \ E = \sum_{j=0}^{j=n} A^j BD^{n-j}.
\]

(24)

Now, for an arbitrary operator \( T, T \in \mathcal{PB}(H) \) if and only if \( T^m \in \mathcal{PB}(H) \) for some \( m \in \mathbb{N} \). This can be easily seen if we observe that for any polynomial \( p \), there exists a unique decomposition of the form

\[
p(z) = p_1(z) + zp_2(z) + z^2 p_3(z) + \ldots + z^{m-1} p_m(z),
\]
where \( p_1, p_2, p_3, \ldots, p_m \) are polynomials in \( z^m \) and \( \| p_j \|_\infty \leq \| p \|_\infty \) for \( j = 1, 2, \ldots, m \). Hence, it suffices to show that \( F \) and therefore \( D \) is polynomially bounded. But now, since for any integer \( k \geq 0 \)

\[
(I - P_{\ker(T^n)})T^sTk(I - P_{\ker(T^n)}) = \begin{bmatrix}
0 & 0 \\
0 & D^{sk}D^k
\end{bmatrix},
\]

which follows from (23) multiplying from the left and from the right by \( I - P_{\ker(T^n)} \), we obtain that \( D \) satisfies the hypothesis of Theorem 1.1. This means that \( D \) is polynomially bounded, and so \( F \) and \( T \) are also.

**Comments.** If we start with a contraction \( T \), let us show that the function \( A_t \) defined in (22) satisfies \( A_t \geq A_s \), for \( 0 \leq t < s \leq 1 \). Indeed, since \( A^2 \geq B^2 \) for positive semidefinite operators implies \( A \geq B \), it is enough to check that \( A_t^2 \geq A_s^2 \) (\( t < s \)). This is equivalent to

\[
(1 - t) \sum_{j=0}^\infty t^j \| T^jh \|^2 \geq (1 - s) \sum_{j=0}^\infty s^j \| T^jh \|^2, \quad h \in \mathcal{H},
\]

and this can be written in the following equivalent form which is clearly true for \( T \) a contraction:

\[
(s - t) \left( \| T^2h \|^2 - \| Th \|^2 \right) + (s^2 - t^2) \left( \| Th \|^2 - \| T^2h \|^2 \right) + (s^3 - t^3) \left( \| T^3h \|^2 - \| T^2h \|^2 \right) + (s^4 - t^4) \left( \| T^4h \|^2 - \| T^3h \|^2 \right) + \ldots \geq 0.
\]

Therefore, it is interesting to ask: what is the class of operators for which the function \( t \to A_t \) is decreasing? In this connection one can easily prove, using ideas similar to those above, the following.

**Theorem 1.6.** Suppose \( T \in \mathcal{PW}(\mathcal{H}) \), the positive-operator-valued function

\[
t \to (1 - t)^{1/2} \left( \sum_{j=0}^\infty t^j T^j \right)^{1/2}, \quad t \in [\varepsilon, 1), \quad 0 < \varepsilon < 1,
\]

is decreasing, and the inequality (5) holds for some positive number \( \alpha \) and a strictly increasing sequence \( \{ t_k \} \) of real numbers converging to 1. Then \( T \) is similar to a contraction.

Another natural question is whether we can weaken the assumption (23) to

\[
(1 - t_k) \sum_{j=0}^\infty t_k^j T^j T^j \geq \alpha(I - P_{\cup_{n=0}^\infty \ker(T^n)}),
\]

and preserve the conclusion in Corollary 1.5. The same counterexample of Foguel in [2] shows that there exists an operator satisfying (25) which is not polynomially bounded.

**Remark.** This paper constitutes part of the author’s Ph.D. thesis written at Texas A&M University under the direction of Professor Carl Pearcy. We thank the referee of this paper who suggested the idea for improving Corollary 1.5.

**Added in proof.** Vern Paulsen has pointed out that the same techniques employed herein can be used to obtain the stronger result that under the hypotheses of Theorem 1.1 or 1.2, \( T \) is completely polynomially bounded, and thus is similar to a contraction.
REFERENCES

1. A. Lebow, A power-bounded operator that is not polynomially bounded, Michigan Math. J. 15 (1968), 397-399. MR 38:5047

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