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GROUP CLOSURES OF ONE-TO-ONE TRANSFORMATIONS

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For a semigroup S of transformations of an infinite set X let G_S be the group of all the permutations of X that preserve S under conjugation. Fix a permutation group H on X and a transformation f of X, and let $\langle f : H\rangle = \langle\{hfh^{-1} : h \in H\}\rangle$ be the H-closure of f. We find necessary and sufficient conditions on a one-to-one transformation f and a normal subgroup H of the symmetric group on X to satisfy $G_{\langle f:H\rangle} = H$. We also show that if S is a semigroup of one-to-one transformations of X and G_S contains the alternating group on X then $\mathrm{Aut}(S) = \mathrm{Inn}(S) \cong G_S$.

1. INTRODUCTION

Given a transformation f of a set X and a group H of permutations of X, the *H-closure* of f in the semigroup $\mathcal{T}_X$ of all the total transformations of X is the semigroup

$$\langle f : H\rangle = \langle\{hfh^{-1} : h \in H\}\rangle.$$

The H-closure of f is the smallest subsemigroup of $\mathcal{T}_X$ that contains f and whose automorphism group $\mathrm{Aut}(\langle f : H\rangle)$ contains all the inner automorphisms $\varphi_h : g \mapsto hgh^{-1}$, where $h \in H$ and $g \in \langle f : H\rangle$.

Let $\mathcal{G}_X$ denote the symmetric group on X. For an arbitrary subsemigroup S of $\mathcal{T}_X$, the group G_S of all the permutations of X that preserve S under conjugation,

$$G_S = \{h \in \mathcal{G}_X : hSh^{-1} \subseteq S\},$$

was introduced in [10]. Given a subgroup H of $\mathcal{G}_X$, a semigroup S of transformations of X is said to be *H-normal* if $G_S = H$. The centraliser $C_{\mathcal{G}_X}(S)$ of S in $\mathcal{G}_X$,

$$C_{\mathcal{G}_X}(S) = \{h \in \mathcal{G}_X : hg = gh, \quad \text{for all } g \in S\},$$

is a normal subgroup of G_S, and the group $\mathrm{Inn}(S)$ of all the inner automorphisms of S is a homomorphic image of G_S, specifically

$$\mathrm{Inn}(S) \cong G_S/C_{\mathcal{G}_X}(S). \tag{1}$$

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While $H \leqslant G_{\langle f:H\rangle}$ for a transformation f of X, the group H may be a proper subgroup of $G_{\langle f:H\rangle}$. For example, if X is a finite set, and f is a total transformation of X having $|X|-1$ elements in its image $\operatorname{im}(f)$, then $G_{\langle f:H\rangle} = \mathcal{G}_X$ precisely when H is a 2-block transitive (see [**11**]). Thus if H is 2-block transitive and f is as stated, then $H = G_{\langle f:H\rangle}$ only when $H = \mathcal{G}_X$. If H is the alternating group $\mathcal{A}_X$ on a finite set X, and f is a non-bijective transformation of X, then $G_{\langle f:H\rangle} = H$ if and only if $|X| \equiv 0 \bmod 4$ and f is an x-nilpotent (see [**6, 7, 8, 9**]). Presently we extend the above studies to the case of an infinite set X by addressing the following problem (see also [**12**]).

PROBLEM 1. *Given an infinite set X, characterise the normal subgroups H of $\mathcal{G}_X$ and transformations f of X such that the semigroup $\langle f : H\rangle$ is H-normal, that is $G_{\langle f:H\rangle} = H$.*

Theorem 4.2 of this paper gives a solution for the above problem when f is a one-to-one transformation. Information on G_S is useful in considering the following problem. A semigroup S of transformations is said to have the *inner automorphism property* [**14**] if all the automorphisms of S are inner.

PROBLEM 2 *Characterise the semigroups of transformations that have the inner automorphism property.*

The automorphisms of specific $\mathcal{G}_X$-normal semigroups were described by a number of authors (see, for example, [**1, 2, 15, 16, 17**]). It was shown in [**18**] (for a finite X) and in [**3**] and [**4**] (for an infinite X) that if S is a $\mathcal{G}_X$-normal semigroup, then S has the inner automorphism property and $\operatorname{Aut}(S) = \operatorname{Inn}(S) \cong \mathcal{G}_X$. If a semigroup of transformations contains certain constant transformations then it has only inner automorphisms [**14**]. If X is finite and S is an $\mathcal{A}_X$-normal semigroup then $\operatorname{Aut}(S) = \operatorname{Inn}(S) \cong \mathcal{A}_X$ [**9**]. If X is finite, and a subgroup H of $\mathcal{G}_X$ is either transitive or equal to its normaliser, then a semigroup S, that is maximal amongst all the H-normal subsemigroups of $\mathcal{T}_X$ containing H, has the inner automorphism property and $\operatorname{Aut}(S) = \operatorname{Inn}(S) \cong H$ [**10**]. Here we continue this line of investigation. We prove that if X is infinite and S is a semigroup of one-to-one transformations such that $\mathcal{A}_X \subseteq G_S$, then S has the inner automorphism property (Theorem 5.5). We also investigate the form of the group $\operatorname{Aut}(S)$.

2. Notation and properties of one-to-one transformations

Let X be an infinite set, and let $\mathcal{W}_X$ be the semigroup of all the total one-to-one transformations of X. There are several parameters associated with a transformation f of X. The *rank* and the *defect* of f are

$$\operatorname{rank}(f) = \left|\operatorname{im}(f)\right|, \text{ and } \operatorname{def}(f) = \left|X - \operatorname{im}(f)\right|.$$

The subset of all the points of X moved by f is

$$S(f) = \left\{x \in X : f(x) \neq x\right\} \text{ and } \operatorname{shift}(f) = \left|S(f)\right|.$$

Just as any permutation of X may be written as a formal product of disjoint finite and infinite cycles, any one-to-one transformation of X may be written (essentially uniquely) as a formal product of disjoint cycles (finite or infinite) and *chains* (defined below) [5]. As usual, transformations f and g are disjoint if $S(f) \cap S(g) = \emptyset$. The formal product of a set A of pairwise disjoint transformations of X is denoted by $\Pi\{f : f \in A\}$ and defined by the following:

$$\Pi\{f : f \in A\}(x) = \begin{cases} f(x), \text{ if } f \in A \text{ and } x \in S(f) \\ x, \text{ if } x \in X - \cup\{S(f) : f \in A\}, \end{cases}$$

where $x \in X$. If $A \subseteq \mathcal{W}_X$ then $\Pi\{f : f \in A\}$ is also in $\mathcal{W}_X$. For a countable ordered subset $Y = \{y_1, y_2, y_3, \dots\}$ of X let $(y_1, y_2, y_3, \dots)$ denote the transformation $f \in \mathcal{W}_X$ such that $f(y_i) = y_{i+1}$ for $i = 1, 2, 3, \dots$, and $f(x) = x$ for all $x \in X - Y$. The transformation $f = (y_1, y_2, y_3, \dots)$ is called a y_1-chain or just a chain. If f is a y_1-chain, then $X - \operatorname{im}(f) = \{y_1\}$ and $\operatorname{def}(f) = 1$. The following result has been proved in [5].

Proposition 2.1. *Let f be a non-identity transformation in $\mathcal{W}_X$. Then f is a formal product of pairwise disjoint cycles and chains, $f = \Pi\{g : g \in A\}$, with no $g \in A$ being a 1-cycle. The number of chains in A is equal to* $\operatorname{def}(f)$. *If $f = \Pi\{g : g \in A'\}$ is another such product then $A = A'$.*

Let $Ch_X \subseteq \mathcal{W}_X$ be the set of all formal products of disjoint chains. Proposition 2.1 assures that every $f \in \mathcal{W}_X$ can be written as a product of two unique disjoint transformations $f_p \in \mathcal{G}_X$ and $f_c \in Ch_X$ (the subscripts p and c stand for permutation and chain correspondingly). The following results are easily derived from elementary properties of one-to-one transformations and an observation that a non-permutation in $\mathcal{W}_X$ has an infinite shift.

Lemma 2.2. *Let $f, g \in \mathcal{W}_X$, then*

1. $\operatorname{def}(fg) = \operatorname{def}(f) + \operatorname{def}(g)$,
2. $\operatorname{shift}(fg) \leqslant \operatorname{shift}(f) + \operatorname{shift}(g)$,
3. *if* $\operatorname{shift}(f) \neq \operatorname{shift}(g)$, *then* $\operatorname{shift}(fg) = \max(\operatorname{shift}(f), \operatorname{shift}(g))$.

For any infinite cardinal α less than or equal to the cardinal successor $|X|^+$ of $|X|$, let

$$S(X, \alpha) = \{f \in \mathcal{G}_X : \operatorname{shift}(f) < \alpha\}.$$

Then $S(X, \alpha)$ is a normal subgroup of the symmetric group $\mathcal{G}_X$ and these groups together with the alternating group $\mathcal{A}_X$ constitute the set of all the non-trivial normal subgroups of $\mathcal{G}_X$ [13].

3. Centralisers of one-to-one transformations

Since the centraliser $C_{\mathcal{G}_X}(S)$ of a semigroup S is a normal subgroup of the group G_S, we start by considering properties of centralisers. For a transformation f of X let

$C_{\mathcal{G}_X}(f) = \{h \in \mathcal{G}_X : hf = fh\}$ be the centraliser of f in $\mathcal{G}_X$. It is self-evident that $C_{\mathcal{G}_X}(f) \leqslant G_{\langle f \rangle}$, and the result below presents a condition sufficient for equality.

PROPOSITION 3.1. *Let f be a one-to-one transformation with a finite defect. Then $G_{\langle f \rangle} = C_{\mathcal{G}_X}(f)$.*

PROOF: Let $h \in G_{\langle f \rangle}$. Then $hfh^{-1} \in \langle f \rangle$, so $hfh^{-1} = f^k$ for some integer $k \geqslant 1$. Therefore $\operatorname{def}(f) = \operatorname{def}(hfh^{-1}) = \operatorname{def}(f^k) = k \operatorname{def}(f)$ by Lemma 2.2. Thus $k = 1$ and $h \in C_{\mathcal{G}_X}(f)$. ∎

Let $N(H)$ denote the normaliser of the group $H \leqslant \mathcal{G}_X$ in $\mathcal{G}_X$. The next result aids in determining the relationship between a normal subgroup H of $\mathcal{G}_X$ and the group $G_{\langle f:H \rangle}$.

LEMMA 3.2. *Let $f \in \mathcal{T}_X$ and $H \leqslant \mathcal{G}_X$. Then $G_{\langle f \rangle} \cap N(H) \leqslant G_{\langle f:H \rangle}$.*

PROOF: Let $h \in G_{\langle f \rangle} \cap N(H)$ and $t \in \langle f : H \rangle$ so that $t = g_1 f {g_1}^{-1} \dots g_n f {g_n}^{-1}$ for some $g_1, \dots, g_n \in H$. Then

$$\begin{aligned} hth^{-1} &= h(g_1 f {g_1}^{-1} \dots g_n f {g_n}^{-1})h^{-1} \\ &= (hg_1h^{-1})(hfh^{-1})(h{g_1}^{-1}h^{-1}) \dots (hg_nh^{-1})(hfh^{-1})(h{g_n}^{-1}h^{-1}) \\ &\in \langle f : H \rangle, \end{aligned}$$

since $hg_i^{\,j}h^{-1} \in H$ for each $i = 1, \dots, n$ and $j = -1$ or 1, so $h \in G_{\langle f:H \rangle}$. ∎

REMARK 3.3. If h and q are permutations of X then hqh^{-1} is a permutation of X that has the same cyclic structure as q. Moreover the permutation hqh^{-1} is obtained by applying h to the symbols in q. Therefore $h \in C_{\mathcal{G}_X}(q)$ precisely when for each (finite or infinite) cycle $(\dots x_i x_{i+1} x_{i+2} \dots)$ of q, the cycle $\big(\dots h(x_i)h(x_{i+1})h(x_{i+2}) \dots\big)$ is also a cycle of q.

Just as the conjugation of permutations preserves their cyclic structure, conjugation of transformations in $\mathcal{W}_X$ by permutations of X preserves the cyclic-chain structure of the transformations [5].

LEMMA 3.4. *Let $f, g \in \mathcal{W}_X$. Then f, g are conjugate if and only if $\operatorname{def}(f) = \operatorname{def}(g)$ and f and g have the same number of cycles of each length (including 1-cycles and infinite cycles) in their cyclic-chain decomposition.*

The next proposition in conjunction with Remark 3.3 describes centralisers of transformations in $\mathcal{W}_X$. For a subset A of X and a permutation h of X, the set $h(A)$ is $\{h(a) : a \in A\}$.

PROPOSITION 3.5. *Take $f \in \mathcal{W}_X$ and write it as a product of disjoint transformations $f = f_p f_c$, where $f_p \in \mathcal{G}_X$, $f_c \in \mathcal{C}h_X$. A permutation $h \in C_{\mathcal{G}_X}(f)$ if and only if*

1. $h \in C_{\mathcal{G}_X}(f_p)$,
2. $h\big(S(f_c)\big) = S(f_c)$, and

3. *for each x_1-chain $(x_1x_2x_3\dots)$ in f_c, $\big(h(x_1)h(x_2)h(x_3)\dots\big)$ is an $h(x_1)$-chain in f_c.*

PROOF: Note that $h \in C_{\mathcal{G}_X}(f)$ if and only if $f_pf_c = hf_ph^{-1}hf_ch^{-1}$, so that by the uniqueness of the cyclic-chain decomposition of f we have that $f_p = hf_ph^{-1}$ and $f_c = hf_ch^{-1}$.

Take a permutation h satisfying conditions (1)–(3) above. Then h commutes with f_p, and we only need to show that $f_c = hf_ch^{-1}$. For any $x \in X - S(f_c)$, we have that $h^{-1}(x) \in X - S(f_c)$, so that $hf_ch^{-1}(x) = hh^{-1}(x) = f_c(x)$. If $x \in S(f_c)$, then $h^{-1}(x) \in S(f_c)$, and there exists a chain $(x_1x_2x_3\dots)$ in f_c such that $h^{-1}(x) = x_i$ for some i. Hence $hf_ch^{-1}(x) = hf_c(x_i) = h(x_{i+1})$, and also $f_c(x) = f_c\big(h(x_i)\big) = h(x_{i+1})$, since $\big(h(x_1)h(x_2)h(x_3)\dots\big)$ is a chain in f_c.

For the converse suppose that $h \in C_{\mathcal{G}_X}(f)$. Then $f_p = hf_ph^{-1}$ implies that condition (1) holds. We show that h maps a chain onto a chain, that is condition (3) holds. Let $(x_1x_2x_3\dots)$ be an x_1-chain in f_c. Since h^{-1} is also in $C_{\mathcal{G}_X}(f)$ we have that $h^{-1}fh(x_i) = f(x_i) = x_{i+1}$, so that $f\big(h(x_i)\big) = h(x_{i+1})$ for each $i = 1, 2, \dots$. Since $x_1 \in X - \mathrm{im}(f) = X - \mathrm{im}(hfh^{-1})$, it follows that $\big(h(x_1)h(x_2)h(x_3)\dots\big)$ is an $h(x_1)$-chain in f_c. Finally condition (2) follows from (3) applied to h (to obtain $h\big(S(f_c)\big) \subseteq S(f_c)$ and h^{-1} (to obtain $h^{-1}\big(S(f_c)\big) \subseteq S(f_c)$, or $h\big(S(f_c)\big) \supseteq S(f_c)$. ∎

The above result has several useful consequences.

COROLLARY 3.6. *Take $f \in \mathcal{W}_X$ and write it as a product of disjoint transformations $f = f_pf_c$, where $f_p \in \mathcal{G}_X$, $f_c \in Ch_X$.*

1. *If $|X - S(f_c)| \leqslant 1$ then the identity permutation i_X is the only element of $C_{\mathcal{G}_X}(f)$ with a finite shift.*
2. *If $\mathrm{def}(f) = 1$ then*
$$C_{\mathcal{G}_X}(f) = \big\{h \in C_{\mathcal{G}_X}(f_p) : h(x) = x \text{ for all } x \in S(f_c)\big\}.$$

PROOF: To prove (1), assume $h \in C_{\mathcal{G}_X}(f)$ is a non-identity permutation. Since $|X - S(f_c)| \leqslant 1$, there exists $x \in S(h) \cap S(f_c)$, so that $x = x_i$ in an x_1-chain $(x_1x_2\dots x_i\dots)$ in f_c. Then by Proposition 3.5, $\big(h(x_1)h(x_2)\dots h(x_i)\dots\big)$ is an $h(x_1)$-chain in f_c. Since $x_i = x \neq h(x) = h(x_i)$, the chains $(x_1x_2\dots x_i\dots)$ and $\big(h(x_1)h(x_2)\dots h(x_i)\dots\big)$ are distinct, so that h maps a countable set $\{x_1, x_2, \dots, x_i, \dots\}$ into its complement in X, therefore $\mathrm{shift}(h)$ is infinite.

To verify (2), note that $\mathrm{def}(f) = 1$ if and only if f_c consists of a single chain. Then by Proposition 3.5, the permutations in $C_{\mathcal{G}_X}(f)$ fix every element of $S(f_c)$ pointwise. ∎

The next result provides necessary conditions for a group $H \leqslant \mathcal{G}_X$ and a transformation $f \in \mathcal{W}_X$ to give rise to an H-normal semigroup $\langle f : H\rangle$.

PROPOSITION 3.7.

1. *Take $f \in \mathcal{W}_X$ and write it as a product of disjoint transformations $f = f_pf_c$, where $f_p \in \mathcal{G}_X$, $f_c \in Ch_X$. Suppose that either*

(a) $\operatorname{def}(f) \geqslant 2$, *or*

(b) $\operatorname{def}(f) = 1$ *and* $|X - S(f_c)| = |X|$.

Then $C_{\mathcal{G}_X}(f)$ *contains a permutation* h *with* $\operatorname{shift}(h) = |X|$.

2. *Take* $f \in \mathcal{W}_X$ *with* $\operatorname{def}(f) \geqslant 2$ *and a group* $H \leqslant S(X, |X|) \trianglelefteq \mathcal{G}_X$.

(c) *If* $C_{\mathcal{G}_X}(f) \leqslant N(H)$ *then* $G_{\langle f:H\rangle} \neq H$.

(d) *If* $H \trianglelefteq \mathcal{G}_X$ *then* $G_{\langle f:H\rangle} \neq H$.

PROOF: We shall concentrate on proving the first result, as the second result is an easy consequence of the first result and Lemma 3.2. Indeed, if h is the permutation as stated in the first result, then, while $h \in C_{\mathcal{G}_X}(f) \cap N(H) \leqslant G_{\langle f\rangle} \cap N(H) \leqslant G_{\langle f:H\rangle}$, we have that h is not an element of H.

Now assume f satisfies the conditions in (1). If $\operatorname{shift}(f_p) = |X|$, then since $f_p \in C_{\mathcal{G}_X}(f)$, we may take $h = f_p$. Thus assume that $\operatorname{shift}(f_p) < |X|$, and so $|X| = |X - S(f_p)| = |X - S(f)| + |S(f_c)|$. Suppose first that $|X - S(f)| = |X|$. Choose a permutation q of $X - S(f)$ that moves every point of $X - S(f)$, and let $h \in \mathcal{G}_X$ coincide with q on $X - S(f)$ and be the identity otherwise. Then $h \in C_{\mathcal{G}_X}(f)$ with $\operatorname{shift}(h) = |X|$, as required.

We may assume now that $|X - S(f)| < |X|$, so that $|X - S(f_c)| = |X - S(f)| + \operatorname{shift}(f_p) < |X|$. Hence by (1b) above, we have that $\operatorname{def}(f) \geqslant 2$, and also $\operatorname{shift}(f_c) = |X|$. Let B be the set of all the chains in f_c, and recall that $|B| = \operatorname{def}(f)$. Take an index set I with $|I| = |B|$ if B is infinite, and $|I| = 1$ if B is finite. Choose $|I|$ disjoint doubleton subsets B_i of B, where $i \in I$, and let $B_i = \{q_i, r_i\}$, where $q_i = (x_1 x_2 \dots), r_i = (y_1 y_2 \dots)$. For each $i \in I$ choose a permutation t_i of X with $S(t_i) = S(q_i) \cup S(r_i)$ that interchanges x_j's and y_j's; that is, for $j = 1, 2, 3, \dots$ we have that $t_i(x_j) = y_j, t_i(y_j) = x_j$ and $t_i(x) = x$ for all $x \in X - (S(q_i) \cup S(r_i))$. Then by Proposition 3.5, each permutation $t_i \in C_{\mathcal{G}_X}(f)$. Observe that the permutations t_i are pairwise disjoint, and take h to be the (formal) product of all t_i's where $i \in I$. By Proposition 3.5 again, the permutation h is in $C_{\mathcal{G}_X}(f)$. Since for each $i \in I$, $\operatorname{shift}(t_i) = \aleph_o$, we have that $\operatorname{shift}(h) = \max(\aleph_o, |I|)$. If $|X| = \aleph_o$, then $\operatorname{shift}(h) = |X|$. If $|X| > \aleph_o$, then since $|X| = \operatorname{shift}(f_c) = \aleph_o |B|$, we have that $|B| = |X|$, so $|I| = |B| = |X|$, and again $\operatorname{shift}(h) = |X|$, as required. ∎

LEMMA 3.8. *Let* Y *be a subset of* X, *and let* q *be a permutation of* Y *having no infinite cycles in its cyclic decomposition. Then* $C_{\mathcal{G}_Y}(q) \cap S(Y, \aleph_o) \leqslant \mathcal{A}_Y$ *if and only if*

1. $|Y - S(q)| \leqslant 1$, *and*
2. q *is a product of disjoint cycles of distinct odd lengths.*

PROOF: Write $q = \Pi\{\alpha_i : i \in I\}$ as a product of disjoint cycles α_i.

Suppose that $C_{\mathcal{G}_Y}(q) \cap S(Y, \aleph_o) \leqslant \mathcal{A}_Y$. Then $|Y - S(q)| \leqslant 1$ (else any 2-cycle (xy) with $x, y \in Y - S(q)$ is an odd permutation in $C_{\mathcal{G}_Y}(q)$). To prove (2), recall that $q = \Pi\{\alpha_i : i \in I\}$, and so for any finite subset J of I, the permutation $q_J = \Pi\{\alpha_i : i \in J\}$ is in $C_{\mathcal{G}_Y}(q)$. By our assumption q_J is an even permutation, therefore each α_i, $i \in I$, has an odd length. If α_i and α_j are two distinct cycles in q of the same odd length, $\alpha_i = (x_1x_2\dots x_k) \neq (y_1y_2\dots y_k) = \alpha_j$, then $t = \Pi\{(x_m, y_m) : m = 1, 2, \dots, k\}$ is an odd finite permutation in $C_{\mathcal{G}_Y}(q)$, a contradiction. Therefore $|\alpha_i| \neq |\alpha_j|$ if $i \neq j$.

Conversely, if q is a product of disjoint cycles α_i of odd distinct length, for $i \in I$, then the group $\langle \alpha_i : i \in I\rangle$, is a subgroup of $C_{\mathcal{G}_Y}(q) \cap \mathcal{A}_Y$. Assume that $|Y - S(q)| \leqslant 1$. We show that, in fact, $\langle \alpha_i : i \in I\rangle = C_{\mathcal{G}_Y}(q) \cap S(Y, \aleph_o)$. Indeed, let $h \in C_{\mathcal{G}_Y}(q) \cap S(Y, \aleph_o)$ and let

$$Z = \{q^k(x) : x \in S(h),\ k \text{ is an integer}\}.$$

Since shift(h) is finite and q is a product of finite cycles, the set Z is finite. Moreover, if $\alpha = (x_1x_2\dots x_m)$ is a cycle in q such that $x_i \in S(h)$, for some $i = 1, 2, \dots, m$, then by the definition of Z, the set $\{x_1, x_2, \dots, x_m\}$ is a subset of Z. If $|X - S(q)| = 1$, then $\{y\} = X - S(q)$ is not Z, since h has to map the single one-cycle (y) onto itself (Observation 3.3). Therefore the restriction $q|_Z$ of q to Z is a permutation of Z that moves every element of Z. Without loss of generality assume that $\alpha_1|_Z, \alpha_2|_Z, \dots, \alpha_n|_Z$ are the restrictions of cycles in q that move the points of Z, and note that $S(\alpha_i) = S(\alpha_i|_Z)$, for $i = 1, 2, \dots, m$. Write $q|_Z = \alpha_1|_Z\alpha_2|_Z\dots\alpha_n|_Z$. Since $S(h) \subseteq Z$, we have that $h|_Z \in C_{\mathcal{G}_Z}(q|_Z)$.

Set $T = \langle \alpha_1|_Z, \alpha_2|_Z, \dots, \alpha_n|_Z\rangle$, and let $\alpha_i|_Z$ be an m_i-cycle, $m_i \geqslant 3$. Then T is a subgroup of $C_{\mathcal{G}_Z}(q|_Z)$ of size $|T| = m_1m_2\dots m_n$. The number of elements in $C_{\mathcal{G}_Z}(q|_Z)$ equals $|\mathcal{G}_Z|$ divided by the number of conjugates of $q|_Z$ in $\mathcal{G}_Z$. Since the number of conjugates of $q|_Z$ in $\mathcal{G}_Z$ equals the number $|Z|!/(m_1!m_2!\dots m_n!)$ of partitions of Z into classes of sizes $m_1, m_2, \dots, m_n$, multiplied by the number $(m_1-1)!(m_2-1)!\dots(m_n-1)!$ of distinct m_i-cycles on the elements of the m_i-class, we see that in fact $T = C_{\mathcal{G}_Z}(q|_Z)$ and $h|_Z \in T$. Therefore $h \in \langle \alpha_i : i \in I\rangle$. ∎

4. H-NORMAL SEMIGROUPS

In this section we characterise those pairs (H, f) of normal subgroups H of the symmetric group $\mathcal{G}_X$ and one-to-one transformations f of X, that produce H-normal semigroups $\langle f : H\rangle$ (having the property that $H = G_{\langle f:H\rangle}$).

LEMMA 4.1. *Let $f \in \mathcal{W}_X$ be a transformation with a finite non-zero defect, and let $H \leqslant \mathcal{G}_X$. Then $G_{\langle f:H\rangle} \leqslant HC_{\mathcal{G}_X}(f)$. If additionally $C_{\mathcal{G}_X}(f) \leqslant N(H)$ then $G_{\langle f:H\rangle} = HC_{\mathcal{G}_X}(f)$.*

PROOF: Take $g \in G_{\langle f:H\rangle}$, then $gfg^{-1} \in \langle f : H\rangle$, and so there exist permutations $q_1, q_2, \dots, q_m \in H$ such that

$$gfg^{-1} = q_1fq_1{}^{-1}q_2fq_2{}^{-1}\dots q_mfq_m{}^{-1}.$$

Then by Lemma 2.2, $\operatorname{def}(f) = \operatorname{def}(gfg^{-1}) = \operatorname{def}(q_1fq_1{}^{-1}) + \operatorname{def}(q_2fq_2{}^{-1}) + \dots + \operatorname{def}(q_mfq_m{}^{-1}) = m\operatorname{def}(f)$, so that $m = 1$. Hence $gfg^{-1} = q_1fq_1{}^{-1}$, and so $q_1{}^{-1}g \in C_{\mathcal{G}_X}(f)$. Therefore $G_{\langle f:H\rangle} \leqslant HC_{\mathcal{G}_X}(f)$.

Now assume that $C_{\mathcal{G}_X}(f) \leqslant N(H)$ and take $h \in H$ and $t \in C_{\mathcal{G}_X}(f)$. Then for any element $q_1 f q_1{}^{-1} q_2 f q_2{}^{-1} \dots q_m f q_m{}^{-1} \in \langle f : H \rangle$ its conjugate by ht is a product of the conjugates of f of the form $htq_i f q_i{}^{-1} t^{-1} h^{-1} = htq_i t^{-1} f t q_i{}^{-1} t^{-1} h^{-1} \in \langle f : H \rangle$ since $tq_i t^{-1} \in H$ for all i. Therefore $ht \in G_{\langle f:H \rangle}$. □

THEOREM 4.2. *Let $f \in \mathcal{W}_X - \mathcal{G}_X$ and write f as a product of disjoint transformations $f = f_p f_c$, where $f_p \in \mathcal{G}_X$, $f_c \in \mathcal{C}h_X$. Take $H \trianglelefteq \mathcal{G}_X$. Then $G_{\langle f:H \rangle} = H$ if and only if one of the following holds:*

1. $H = \mathcal{G}_X$,
2. $H = S(X, \aleph_o)$, $|X| = \aleph_o$, $\operatorname{def}(f) = 1$, $|X - S(f_c)| < \aleph_o$,
3. $H = \mathcal{A}_X$, $|X| = \aleph_o$, $\operatorname{def}(f) = 1$, $|X - S(f_c)| < \aleph_o$, $|X - S(f)| \leqslant 1$, *and f_p is a product of disjoint cycles of distinct odd lengths,*
4. $H = \{i_X\}$, $|X| = \aleph_o$, $\operatorname{def}(f) = 1$, $|X - S(f_c)| \leqslant 1$.

PROOF: Suppose that H is a proper normal subgroup of $\mathcal{G}_X$, so that $H \leqslant S(X, |X|)$, and assume that $H = G_{\langle f:H \rangle}$ for a one-to one transformation f. By Proposition 3.7, we have that $\operatorname{def}(f) = 1$ and $|X - S(f_c)| < |X|$ so that $|S(f_c)| = |X|$. Since the defect of f is 1, f_c consists of a single chain, and so $|S(f_c)| = \aleph_o$. Therefore X is countable and $X - S(f_c)$ is at most finite. By Lemma 4.1 we have that $H = G_{\langle f:H \rangle} = HC_{\mathcal{G}_X}(f)$, so by Corollary 3.6,

$$\{h \in C_{\mathcal{G}_X}(f_p) : h(x) = x \text{ for all } x \in S(f_c)\} = C_{\mathcal{G}_X}(f) \leqslant H.$$

When X is countable the only non-trivial proper normal subgroups of $\mathcal{G}_X$ are $S(X, \aleph_o)$ and $\mathcal{A}_X$. If $H = \mathcal{A}_X$, then it follows from Lemma 3.8 that f can fix at most one point of X and f_p is a product of disjoint cycles of distinct odd lengths. If $H = \{i_X\}$ then $C_{\mathcal{G}_X}(f) = \{i_X\}$ so that $C_{\mathcal{G}_X}(f_p) = \{i_X\}$ and hence $|X - S(f_c)| \leqslant 1$.

For the converse note that $H \leqslant G_{\langle f:H \rangle} \leqslant \mathcal{G}_X$ for any subgroup H of $\mathcal{G}_X$, therefore if $H = \mathcal{G}_X$ we have that $G_{\langle f:\mathcal{G}_X \rangle} = \mathcal{G}_X$. Now assume that $H \leqslant S(X, \aleph_o)$, X is countable, $\operatorname{def}(f) = 1$ and $X - S(f_c)$ is finite. Then by Lemma 4.1 and Corollary 3.6, we have that $G_{\langle f:H \rangle} = HC_{\mathcal{G}_X}(f) = H\{h \in C_{\mathcal{G}_X}(f_p) : h(x) = x \text{ for all } x \in S(f_c)\}$. Since $X - S(f_c)$ is finite, $C_{\mathcal{G}_X}(f) \leqslant S(X, \aleph_o)$, so $G_{\langle f:H \rangle} = H$ for $H = S(X, \aleph_o)$.

If we assume additionally that f_p is a product of disjoint cycles of distinct odd lengths, and f fixes at most one point, then Corollary 3.6 and Lemma 3.8 imply that $C_{\mathcal{G}_X}(f) \leqslant \mathcal{A}_X$, and so $G_{\langle f:\mathcal{A}_X \rangle} = \mathcal{A}_X$. Similarly, if $|X - S(f_c)| \leqslant 1$, then $C_{\mathcal{G}_X}(f) = \{i_X\}$, and $G_{\langle f:\{i_X\} \rangle} = \{i_X\}$. □

5. AUTOMORPHISMS

If S is a semigroup of total transformations of a *finite* set X, and G_S contains the alternating group $\mathcal{A}_X$ on X, then $G_S = \mathcal{G}_X$, S is a $\mathcal{G}_X$-normal semigroup, all the automorphisms of S are inner, and the automorphism group $\mathrm{Aut}(S)$ of S is isomorphic to $\mathcal{G}_X$ [6]. For an infinite set X the fact that $\mathcal{A}_X \leqslant G_S$ does not imply that $G_S = \mathcal{G}_X$. However it will be shown in this section that if $S \not\subseteq \mathcal{G}_X$ is a semigroup of one-to-one transformations of an infinite set X such that G_S contains $\mathcal{A}_X$, then S has the inner automorphism property. The technique used here is based on that of [3] developed to describe the automorphisms of $\mathcal{G}_X$ -normal semigroups.

Everywhere in this section we assume that S is a subsemigroup of $\mathcal{W}_X$ that contains transformations with non-zero defects, and that $\mathcal{A}_X \leqslant G_S$. To describe the automorphism group $\mathrm{Aut}(S)$, in view of Equation 1 (in Section 1), we need to know the structure of the centraliser of the semigroup S in $\mathcal{G}_X$.

Proposition 5.1. *The centraliser $C_{\mathcal{G}_X}(S)$ of S is equal to $\{i_X\}$.*

Proof: Let $f \in S$ and let $T = \langle f : \mathcal{A}_X\rangle$ be a subsemigroup of S. We show that $C_{\mathcal{G}_X}(T) = \{i_X\}$, and deduce the statement of the Proposition from an observation that since T is a subsemigroup of S, the centraliser $C_{\mathcal{G}_X}(S) \subseteq C_{\mathcal{G}_X}(T)$. First we demonstrate that

$$C_{\mathcal{G}_X}(T) = \cap\{hC_{\mathcal{G}_X}(f)h^{-1} : h \in \mathcal{A}_X\}. \tag{2}$$

Indeed for each $q \in C_{\mathcal{G}_X}(T)$ and $h \in \mathcal{A}_X$ we have that $qhfh^{-1}q^{-1} = hfh^{-1}$, so that $h^{-1}qh \in C_{\mathcal{G}_X}(f)$, and $q \in h\,C_{\mathcal{G}_X}(f)\,h^{-1}$. Conversely, assume that $p \in \cap\{hC_{\mathcal{G}_X}(f)h^{-1} : h \in \mathcal{A}_X\}$ and take $g = h_1fh_1^{-1}h_2fh_2^{-1}\dots h_mfh_m^{-1} \in T$. For each $i = 1, 2, \dots, m$, there exists $r_i \in C_{\mathcal{G}_X}(f)$ such that $p = h_ir_ih_i^{-1}$. Therefore $ph_ifh_i^{-1}p^{-1} = h_ir_ih_i^{-1}h_ifh_i^{-1}h_ir_i{}^{-1}h_i^{-1} = h_ifh_i^{-1}$, so that $pgp^{-1} = g$, and $g \in C_{\mathcal{G}_X}(T)$.

Take $g \in C_{\mathcal{G}_X}(T) \subseteq C_{\mathcal{G}_X}(f)$, and suppose that g maps a chain $(x_1x_2x_3\dots)$ of f to a different chain $(g(x_1)g(x_2)g(x_3)\dots)$ of f (Proposition 3.5). Take $s = (x_1x_2x_3) \in \mathcal{A}_X$. By Equation (2) above, $g = sqs^{-1}$ for some $q \in C_{\mathcal{G}_X}(f)$, and this q has to map every chain of f onto a chain in f in prescribed order (Proposition 3.5). However we have that $q(x_1) = s^{-1}gs(x_1) = s^{-1}g(x_2) = g(x_2) = x_3$, hence $q(x_1qx_2x_3\dots)$ is not a chain in f. This contradiction proves that g fixes every point of $S(f_c)$.

Suppose now that there is an $x \in X - S(f_c)$ such that $g(x) = y \neq x$, and note that $y \in X - S(f_c)$. Choose $z \in S(f_c)$ and take $s_1 = (xyz) \in \mathcal{A}_X$. By Equation (2) again, $g = s_1q_1s_1^{-1}$ for some $q_1 \in C_{\mathcal{G}_X}(f)$. However, in this case $q_1(z) = s_1^{-1}gs_1(z) = s_1^{-1}g(x) = s_1^{-1}(y) = x$, so $q_1\big(S(f_c)\big) \neq S(f_c)$, a contradiction to the fact that $q_1 \in C_{\mathcal{G}_X}(f)$ (Proposition 3.5 again). Therefore g is the identity permutation of X. ∎

We proceed with the description of $\mathrm{Aut}(S)$. For an $x \in X$ define

$$\mathcal{R}_x = \{r \in S : x \in X - \mathrm{im}(r)\}.$$

In as much as G_S contains a transitive group $\mathcal{A}_X$, the set $\mathcal{R}_x$ is non-empty for every $x \in X$. In fact $\mathcal{R}_x$ is a right ideal of S, termed a *point right ideal*. Moreover, for any distinct points $x, y \in X$, the corresponding point right ideals $\mathcal{R}_x$ and $\mathcal{R}_y$ are also distinct. Indeed if $r \in \mathcal{R}_x \cap \mathcal{R}_y$, choose distinct points $u, v \in \operatorname{im}(r)$ and take $h = (yuv)$ to be a three-cycle in $\mathcal{A}_X \leqslant G_S$. Then $\operatorname{im}(hrh^{-1}) = h(\operatorname{im}(r))$, and so $hrh^{-1} \in \mathcal{R}_x - \mathcal{R}_y$. Therefore there is a one-to-one correspondence between the points x of X and the point right ideals $\mathcal{R}_x$ of S.

We show that any automorphism of S acts faithfully on the set $\{\mathcal{R}_x : x \in X\}$ of all the point right ideals of S. Given distinct transformations s and t in S, define

$$\mathcal{R}(s,t) = \{r \in S : sr = tr\}.$$

If non-empty, $\mathcal{R}(s,t)$ is a right ideal of S termed a *function right ideal.* It is not difficult to see that there is a relationship between non-empty function right ideals and point right ideals of S (see [**3**]) given by

$$\mathcal{R}(s,t) = \cap\{\mathcal{R}_x : s(x) \neq t(x)\}. \tag{3}$$

LEMMA 5.2. *For each $x \in X$ there exist transformations $s, t \in S$ such that $\mathcal{R}_x = \mathcal{R}(s,t)$.*

PROOF: Since the defect of a product of two one-to-one transformations is the sum of their defects, and since S contains transformations with non-zero defects, we may choose a transformation g in S with $\operatorname{def}(g) \geqslant 3$. Since G_S contains a transitive group $\mathcal{A}_X$ we may assume without loss of generality that $x \in X - \operatorname{im}(g)$. Let $g(x) = y$, and choose two other distinct points u and z in $X - \operatorname{im}(g)$. Take three-cycles $h_1 = (xzu)$ and $h_2 = (xzy)$ in $\mathcal{A}_X \leqslant G_S$, and let $s = h_1 g {h_1}^{-1} g$ and $t = h_2 g {h_2}^{-1} g$.

We show that the above defined s and t are the required transformations. Indeed, $s(x) = h_1 g {h_1}^{-1} g(x) = h_1 g {h_1}^{-1}(y) = h_1 g(y) = g(y)$, since $g(y)$ is not an element of $\{x, u, z\} \subseteq X - \operatorname{im}(f)$. Also $t(x) = h_2 g {h_2}^{-1} g(x) = h_2 g {h_2}^{-1}(y) = h_2 g(z) = g(z)$, since $g(z) \neq g(x) = y$, and $g(z) \neq x, z \in X - \operatorname{im}(g)$, therefore $s(x) \neq t(x)$. If $a \neq x$, then $g(a) \notin \{x, y, u, z\}$, so $h_1^{-1} g(a) = h_2^{-1} g(a) \notin \{x, y, u, z\}$, and it is easy to see that $s(a) = t(a)$. ∎

The set of function right ideals is partially ordered by set inclusion, and its maximal elements are of the form $\mathcal{R}(s,t)$ where s and t differ precisely on one point of X (Equation 3 and Lemma 5.2). Formally:

LEMMA 5.3. *Given transformations $s, t \in S$, $\mathcal{R}(s,t)$ is a maximal function right ideal of S if and only if $\mathcal{R}(s,t) = \mathcal{R}_x$, for some $x \in X$.*

Take an automorphism φ of S and observe that φ acts on the set of function right ideals:

$$\begin{aligned} \varphi(\mathcal{R}(s,t)) &= \{\varphi(r) : r \in S,\ \varphi(sr) = \varphi(tr)\} \\ &= \{r' : r' \in S,\ \varphi(s)r' = \varphi(t)r'\} \\ &= \mathcal{R}(\varphi(s),\ \varphi(t)). \end{aligned}$$

Moreover φ maps the set of all maximal function right ideals onto itself, hereby giving rise to a permutation h of X such that for an $x \in X$, $h(x) = y$ if $\varphi(\mathcal{R}_x) = \mathcal{R}_y$ (Lemma 5.3). The next result follows then from the observation that for any $x \in X$ and $f \in S$ we have that $x \in X - \operatorname{im}(f)$ if and only if $f \in \mathcal{R}_x$ if and only if $\varphi(f) \in \varphi(\mathcal{R}_x) = \mathcal{R}_{h(x)}$.

LEMMA 5.4. *Given $f \in S$, $\operatorname{im}(\varphi(f)) = h(\operatorname{im}(f))$.*

To see that φ indeed acts on S by conjugation by h, take an arbitrary $x \in X$, $f \in S$, and choose a non-permutation g in S with $x \in \operatorname{im}(g)$. Take $u \in \operatorname{im}(g)$ with $u \neq x$ and $v \in X - \operatorname{im}(g)$, and let $q = (uxv) \in \mathcal{A}_X \leqslant G_S$. Then $qgq^{-1} \in S$ and $\operatorname{im}(qgq^{-1}) = q(\operatorname{im}(g)) = \operatorname{im}(g) - \{x\} \cup \{v\}$, so that $\operatorname{im}(g) - \operatorname{im}(qgq^{-1}) = \{x\}$. By Lemma 5.4,

$$\begin{aligned}\varphi(f)(h(x)) &= \varphi(f)(\operatorname{im}(\varphi(g)) - \operatorname{im}(\varphi(qgq^{-1}))\\ &= \operatorname{im}(\varphi(fg)) - \operatorname{im}(\varphi(fqgq^{-1}))\\ &= hf(x),\end{aligned}$$

and so $\varphi(f) = hfh^{-1}$. The above discussion together with Proposition 5.1 implies the next result.

THEOREM 5.5. *Let X be an infinite set, and let S be a semigroup of one-to-one transformations of X that contains non-permutations. If the alternating group $\mathcal{A}_X$ is a subgroup of G_S, then each automorphism φ of S is inner, and $\operatorname{Aut}(S) \cong G_S$.*

COROLLARY 5.6. *Let $f \in \mathcal{W}_X$ be a transformation with a non-zero defect, and let H be a normal subgroup of $\mathcal{G}_X$, then*

1. $\operatorname{Aut}(\langle f : H\rangle) = \operatorname{Inn}(\langle f : H\rangle)$,
2. *if $H \neq \{i_X\}$ and f has a finite defect, then*
$$\operatorname{Aut}(\langle f : H\rangle) = \operatorname{Inn}(\langle f : H\rangle) \cong HC_{\mathcal{G}_X}(f).$$

PROOF: To prove the first part of the Corollary, note that if H is a non-trivial normal subgroup of $\mathcal{G}_X$, then the result follows from Theorem 5.5. If $H = \{i_X\}$, then $\langle f : H\rangle$ is the monogenic semigroup generated by f. Since $f \in \mathcal{W}_X - \mathcal{G}_X$, for any integer $k \geqslant 2$ we have that $f^k \neq f$ and so the identity automorphism is the only automorphism of $\langle f : H\rangle$.

The second part of the Corollary follows directly from Theorem 5.5 and Lemma 4.1. ∎

Observe that if H is a proper normal subgroup of $\mathcal{G}_X$ and $f \in \mathcal{W}_X$ is a non-permutation satisfying $\operatorname{Aut}(\langle f : H\rangle) \cong H$, then by Proposition 3.7 and Corollary 5.6, we have that $\operatorname{def}(f) = 1$ and $|X - S(f_c)| < |X|$, so that X is a countable set.

COROLLARY 5.7. *Let X be a countable set. Then there exists a non-permutation $f \in \mathcal{W}_X$ such that for any normal subgroup H of $\mathcal{G}_X$ we have that $\operatorname{Aut}(\langle f : H\rangle) \cong H$.*

PROOF: Take f to be a single chain shifting all the points of X. Then, by Corollary 3.6, $C_{\mathcal{G}_X}(f) = \{i_X\}$. The result follows from Corollary 5.6. □

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