GROUP CLOSURES OF ONE-TO-ONE TRANSFORMATIONS

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For a semigroup $S$ of transformations of an infinite set $X$ let $G_S$ be the group of all the permutations of $X$ that preserve $S$ under conjugation. Fix a permutation group $H$ on $X$ and a transformation $f$ of $X$, and let $(f : H) = \langle \{hfh^{-1} : h \in H\} \rangle$ be the $H$-closure of $f$. We find necessary and sufficient conditions on a one-to-one transformation $f$ and a normal subgroup $H$ of the symmetric group on $X$ to satisfy $G_{(f : H)} = H$. We also show that if $S$ is a semigroup of one-to-one transformations of $X$ and $G_S$ contains the alternating group on $X$ then $\text{Aut}(S) = \text{Inn}(S) \cong G_S$.

1. INTRODUCTION

Given a transformation $f$ of a set $X$ and a group $H$ of permutations of $X$, the $H$-closure of $f$ in the semigroup $T_X$ of all the total transformations of $X$ is the semigroup

$$\langle f : H \rangle = \langle \{hfh^{-1} : h \in H\} \rangle.$$ 

The $H$-closure of $f$ is the smallest subsemigroup of $T_X$ that contains $f$ and whose automorphism group $\text{Aut}(\langle f : H \rangle)$ contains all the inner automorphisms $\varphi_h : g \mapsto hgh^{-1}$, where $h \in H$ and $g \in \langle f : H \rangle$.

Let $G_X$ denote the symmetric group on $X$. For an arbitrary subsemigroup $S$ of $T_X$, the group $G_S$ of all the permutations of $X$ that preserve $S$ under conjugation,

$$G_S = \{ h \in G_X : hSh^{-1} \subseteq S \},$$

was introduced in [10]. Given a subgroup $H$ of $G_X$, a semigroup $S$ of transformations of $X$ is said to be $H$-normal if $G_S = H$. The centraliser $C_{g_X}(S)$ of $S$ in $G_X$,

$$C_{g_X}(S) = \{ h \in G_X : hg = gh, \text{ for all } g \in S \},$$

is a normal subgroup of $G_S$, and the group $\text{Inn}(S)$ of all the inner automorphisms of $S$ is a homomorphic image of $G_S$, specifically

$$(1) \quad \text{Inn}(S) \cong G_S/C_{g_X}(S).$$
While $H \leq G_{(f:H)}$ for a transformation $f$ of $X$, the group $H$ may be a proper subgroup of $G_{(f:H)}$. For example, if $X$ is a finite set, and $f$ is a total transformation of $X$ having $|X| - 1$ elements in its image $\text{im}(f)$, then $G_{(f:H)} = G_X$ precisely when $H$ is a 2-block transitive (see [11]). Thus if $H$ is 2-block transitive and $f$ is as stated, then $H = G_{(f:H)}$ only when $H = G_X$. If $H$ is the alternating group $A_X$ on a finite set $X$, and $f$ is a non-bijective transformation of $X$, then $G_{(f:H)} = H$ if and only if $|X| \equiv 0 \mod 4$ and $f$ is an $x$-nilpotent (see [6, 7, 8, 9]). Presently we extend the above studies to the case of an infinite set $X$ by addressing the following problem (see also [12]).

**Problem 1.** Given an infinite set $X$, characterise the normal subgroups $H$ of $G_X$ and transformations $f$ of $X$ such that the semigroup $(f : H)$ is $H$-normal, that is $G_{(f:H)} = H$.

Theorem 4.2 of this paper gives a solution for the above problem when $f$ is a one-to-one transformation. Information on $G_S$ is useful in considering the following problem. A semigroup $S$ of transformations is said to have the inner automorphism property [14] if all the automorphisms of $S$ are inner.

**Problem 2** Characterise the semigroups of transformations that have the inner automorphism property.

The automorphisms of specific $G_X$-normal semigroups were described by a number of authors (see, for example, [1, 2, 15, 16, 17]). It was shown in [18] (for a finite $X$) and in [3] and [4] (for an infinite $X$) that if $S$ is a $G_X$-normal semigroup, then $S$ has the inner automorphism property and $\text{Aut}(S) = \text{Inn}(S) \cong G_X$. If a semigroup of transformations contains certain constant transformations then it has only inner automorphisms [14]. If $X$ is finite and $S$ is an $A_X$-normal semigroup then $\text{Aut}(S) = \text{Inn}(S) \cong A_X$ [9]. If $X$ is finite, and a subgroup $H$ of $G_X$ is either transitive or equal to its normaliser, then a semigroup $S$, that is maximal amongst all the $H$-normal subsemigroups of $T_X$ containing $H$, has the inner automorphism property and $\text{Aut}(S) = \text{Inn}(S) \cong H$ [10]. Here we continue this line of investigation. We prove that if $X$ is infinite and $S$ is a semigroup of one-to-one transformations such that $A_X \subseteq G_S$, then $S$ has the inner automorphism property (Theorem 5.5). We also investigate the form of the group $\text{Aut}(S)$.

2. **Notation and Properties of One-to-One Transformations**

Let $X$ be an infinite set, and let $\mathcal{W}_X$ be the semigroup of all the total one-to-one transformations of $X$. There are several parameters associated with a transformation $f$ of $X$. The rank and the defect of $f$ are

\[
\text{rank}(f) = |\text{im}(f)|, \quad \text{and def}(f) = |X - \text{im}(f)|.
\]

The subset of all the points of $X$ moved by $f$ is

\[
S(f) = \{x \in X : f(x) \neq x\} \quad \text{and shift}(f) = |S(f)|.
\]
Just as any permutation of \( X \) may be written as a formal product of disjoint finite and infinite cycles, any one-to-one transformation of \( X \) may be written (essentially uniquely) as a formal product of disjoint cycles (finite or infinite) and \textit{chains} (defined below) [5]. As usual, transformations \( f \) and \( g \) are disjoint if \( S(f) \cap S(g) = \emptyset \). The formal product of a set \( A \) of pairwise disjoint transformations of \( X \) is denoted by \( \prod \{ f : f \in A \} \) and defined by the following:

\[
\prod \{ f : f \in A \} (x) = \begin{cases} 
  f(x), & \text{if } f \in A \text{ and } x \in S(f) \\
  x, & \text{if } x \in X - \bigcup \{ S(f) : f \in A \},
\end{cases}
\]

where \( x \in X \). If \( A \subseteq \mathcal{W}_X \) then \( \prod \{ f : f \in A \} \) is also in \( \mathcal{W}_X \). For a countable ordered subset \( Y = \{ y_1, y_2, y_3, \ldots \} \) of \( X \) let \( (y_1, y_2, y_3, \ldots) \) denote the transformation \( f \in \mathcal{W}_X \) such that \( f(y_i) = y_{i+1} \) for \( i = 1, 2, 3, \ldots \), and \( f(x) = x \) for all \( x \in X - Y \). The transformation \( f = (y_1, y_2, y_3, \ldots) \) is called a \( y_1 \)-chain or just a chain. If \( f \) is a \( y_1 \)-chain, then \( X - \text{im}(f) = \{ y_i \} \) and \( \text{def}(f) = 1 \). The following result has been proved in [5].

\begin{proposition}
Let \( f \) be a non-identity transformation in \( \mathcal{W}_X \). Then \( f \) is a formal product of pairwise disjoint cycles and chains, \( f = \prod \{ g : g \in A \} \), with no \( g \in A \) being a 1-cycle. The number of chains in \( A \) is equal to \( \text{def}(f) \). If \( f = \prod \{ g : g \in A' \} \) is another such product then \( A = A' \).
\end{proposition}

Let \( \text{Ch}_X \subseteq \mathcal{W}_X \) be the set of all formal products of disjoint chains. Proposition 2.1 assures that every \( f \in \mathcal{W}_X \) can be written as a product of two unique disjoint transformations \( f_p \in \mathcal{G}_X \) and \( f_c \in \text{Ch}_X \) (the subscripts \( p \) and \( c \) stand for permutation and chain correspondingly). The following results are easily derived from elementary properties of one-to-one transformations and an observation that a non-permutation in \( \mathcal{W}_X \) has an infinite shift.

\begin{lemma}
Let \( f, g \in \mathcal{W}_X \), then
\begin{enumerate}
  \item \( \text{def}(fg) = \text{def}(f) + \text{def}(g) \),
  \item \( \text{shift}(fg) \leq \text{shift}(f) + \text{shift}(g) \),
  \item if \( \text{shift}(f) \neq \text{shift}(g) \), then \( \text{shift}(fg) = \max(\text{shift}(f), \text{shift}(g)) \).
\end{enumerate}
\end{lemma}

For any infinite cardinal \( \alpha \) less than or equal to the cardinal successor \( |X|^+ \) of \( |X| \), let

\[
S(X, \alpha) = \{ f \in \mathcal{G}_X : \text{shift}(f) < \alpha \}.
\]

Then \( S(X, \alpha) \) is a normal subgroup of the symmetric group \( \mathcal{G}_X \) and these groups together with the alternating group \( \mathcal{A}_X \) constitute the set of all the non-trivial normal subgroups of \( \mathcal{G}_X \) [13].

3. CENTRALISERS OF ONE-TO-ONE TRANSFORMATIONS

Since the centraliser \( \mathcal{G}_X(S) \) of a semigroup \( S \) is a normal subgroup of the group \( \mathcal{G}_S \), we start by considering properties of centralisers. For a transformation \( f \) of \( X \) let
$C_{G_X}(f) = \{ h \in G_X : hf = fh \}$ be the centraliser of $f$ in $G_X$. It is self-evident that $C_{G_X}(f) \subseteq G(f)$, and the result below presents a condition sufficient for equality.

**Proposition 3.1.** Let $f$ be a one-to-one transformation with a finite defect. Then $G(f) = C_{G_X}(f)$.

**Proof:** Let $h \in G(f)$. Then $hf^{-1} \in \langle f \rangle$, so $hf^{-1} = f^k$ for some integer $k \geq 1$. Therefore $\text{def}(f) = \text{def}(hf^{-1}) = \text{def}(f^k) = k \text{def}(f)$ by Lemma 2.2. Thus $k = 1$ and $h \in C_{G_X}(f)$. \hfill $\square$

Let $N(H)$ denote the normaliser of the group $H \leq G_X$ in $G_X$. The next result aids in determining the relationship between a normal subgroup $H$ of $G_X$ and the group $G(f; H)$.

**Lemma 3.2.** Let $f \in T_X$ and $H \leq G_X$. Then $G(f) \cap N(H) \leq G(f; H)$.

**Proof:** Let $h \in G(f) \cap N(H)$ and $t \in \langle f : H \rangle$ so that $t = g_1f g_1^{-1} \cdots g_n f g_n^{-1}$ for some $g_1, \ldots, g_n \in H$. Then

$$hth^{-1} = h(g_1 f g_1^{-1} \cdots g_n f g_n^{-1})h^{-1} = (h g_1 h^{-1})(h f h^{-1})(h g_1^{-1} h^{-1}) \cdots (h g_n h^{-1})(h f h^{-1})(h g_n^{-1} h^{-1})$$

since $h g_i h^{-1} \in H$ for each $i = 1, \ldots, n$ and $j = -1$ or 1, so $h \in G(f; H)$. \hfill $\square$

**Remark 3.3.** If $h$ and $q$ are permutations of $X$ then $hqh^{-1}$ is a permutation of $X$ that has the same cyclic structure as $q$. Moreover the permutation $hqh^{-1}$ is obtained by applying $h$ to the symbols in $q$. Therefore $h \in C_{G_X}(q)$ precisely when for each (finite or infinite) cycle $(\ldots x_i x_{i+1} x_{i+2} \ldots)$ of $q$, the cycle $(\ldots h(x_i) h(x_{i+1}) h(x_{i+2}) \ldots)$ is also a cycle of $q$.

Just as the conjugation of permutations preserves their cyclic structure, conjugation of transformations in $W_X$ by permutations of $X$ preserves the cyclic-chain structure of the transformations [5].

**Lemma 3.4.** Let $f, g \in W_X$. Then $f, g$ are conjugate if and only if $\text{def}(f) = \text{def}(g)$ and $f$ and $g$ have the same number of cycles of each length (including 1-cycles and infinite cycles) in their cyclic-chain decomposition.

The next proposition in conjunction with Remark 3.3 describes centralisers of transformations in $W_X$. For a subset $A$ of $X$ and a permutation $h$ of $X$, the set $h(A)$ is $\{ h(a) : a \in A \}$.

**Proposition 3.5.** Take $f \in W_X$ and write it as a product of disjoint transformations $f = f_p f_c$, where $f_p \in G_X$, $f_c \in C_{G_X}(f)$. A permutation $h \in C_{G_X}(f)$ if and only if

1. $h \in C_{G_X}(f_p)$,
2. $h(S(f_c)) = S(f_c)$, and
3. for each $x_1$-chain $(x_1x_2x_3\ldots)$ in $f_c$, $(h(x_1)h(x_2)h(x_3)\ldots)$ is an $h(x_1)$-chain in $f_c$.

**PROOF:** Note that $h \in C_{g\chi}(f)$ if and only if $f_pf_c = hf_ph^{-1}h_fh^{-1}$, so that by the uniqueness of the cyclic-chain decomposition of $f$ we have that $f_p = hf_ph^{-1}$ and $f_c = hfh^{-1}$.

Take a permutation $h$ satisfying conditions (1)–(3) above. Then $h$ commutes with $f_p$, and we only need to show that $f_c = hf_fh^{-1}$. For any $x \in X - S(f_c)$, we have that $h^{-1}(x) \in X - S(f_c)$, so that $hf_fh^{-1}(x) = hh^{-1}(x) = f_c(x)$. If $x \in S(f_c)$, then $h^{-1}(x) \in S(f_c)$, and there exists a chain $(x_1x_2x_3\ldots)$ in $f_c$ such that $h^{-1}(x) = x_i$ for some $i$. Hence $hfh^{-1}(x) = hf_c(x_i) = h(x_{i+1})$, and also $f_c(x) = f_c(h(x_i)) = h(x_{i+1})$, since $(h(x_1)h(x_2)h(x_3)\ldots)$ is a chain in $f_c$.

For the converse suppose that $h \in C_{g\chi}(f)$. Then $f_p = hf_ph^{-1}$ implies that condition (1) holds. We show that $h$ maps a chain onto a chain, that is condition (3) holds. Let $(x_1x_2x_3\ldots)$ be an $x_1$-chain in $f_c$. Since $h^{-1}$ is also in $C_{g\chi}(f)$ we have that $h^{-1}fh(x_i) = f(x_i) = x_{i+1}$, so that $f(h(x_i)) = h(x_{i+1})$ for each $i = 1, 2, \ldots$. Since $x_1 \in X - \text{im}(f) = X - \text{im}(hfh^{-1})$, it follows that $(h(x_1)h(x_2)h(x_3)\ldots)$ is an $h(x_1)$-chain in $f_c$. Finally condition (2) follows from (3) applied to $h$ (to obtain $h(S(f_c)) \subseteq S(f_c)$ and $h^{-1}$ (to obtain $h^{-1}(S(f_c)) \subseteq S(f_c)$, or $h(S(f_c)) \supseteq S(f_c)$).

The above result has several useful consequences.

**COROLLARY 3.6.** Take $f \in W_X$ and write it as a product of disjoint transformations $f = f_pf_c$, where $f_p \in G_X$, $f_c \in Ch_X$.

1. If $|X - S(f_c)| \leq 1$ then the identity permutation $i_X$ is the only element of $C_{g\chi}(f)$ with a finite shift.
2. If $\text{def}(f) = 1$ then $C_{g\chi}(f) = \{ h \in C_{g\chi}(f_p) : h(x) = x \text{ for all } x \in S(f_c) \}$.

**PROOF:** To prove (1), assume $h \in C_{g\chi}(f)$ is a non-identity permutation. Since $|X - S(f_c)| \leq 1$, there exists $x \in S(h) \cap S(f_c)$, so that $x = x_i$ in an $x_1$-chain $(x_1x_2\ldots x_i\ldots)$ in $f_c$. Then by Proposition 3.5, $(h(x_1)h(x_2)\ldots h(x_i)\ldots)$ is an $h(x_1)$-chain in $f_c$. Since $x_i = x \neq h(x) = h(x_i)$, the chains $(x_1x_2\ldots x_i\ldots)$ and $(h(x_1)h(x_2)\ldots h(x_i)\ldots)$ are distinct, so that $h$ maps a countable set $\{x_1, x_2, \ldots, x_i, \ldots\}$ into its complement in $X$, therefore shift$(h)$ is infinite.

To verify (2), note that def$(f) = 1$ if and only if $f_c$ consists of a single chain. Then by Proposition 3.5, the permutations in $C_{g\chi}(f)$ fix every element of $S(f_c)$ pointwise.

The next result provides necessary conditions for a group $H \leq G_X$ and a transformation $f \in W_X$ to give rise to an $H$-normal semigroup $(f : H)$.

**PROPOSITION 3.7.**

1. Take $f \in W_X$ and write it as a product of disjoint transformations $f = f_pf_c$, where $f_p \in G_X$, $f_c \in Ch_X$. Suppose that either
1. \( \text{def}(f) \geq 2 \), or
2. \( \text{def}(f) = 1 \) and \( |X - S(f_a)| = |X| \).

Then \( C_G(f) \) contains a permutation \( h \) with \( \text{shift}(h) = |X| \).

2. Take \( f \in \mathcal{W}_X \) with \( \text{def}(f) > 2 \) and a group \( H \leq S(X, |X|) \subseteq G_X \).

(c) If \( C_G(f) \leq N(H) \) then \( G(f, H) \neq H \).

d) If \( H \leq G_X \) then \( G(f, H) \neq H \).

**Proof:** We shall concentrate on proving the first result, as the second result is an easy consequence of the first result and Lemma 3.2. Indeed, if \( h \) is the permutation as stated in the first result, then, while \( h \in C_G(f) \cap N(H) \subseteq G(f) \cap N(H) \subseteq G(f, H) \), we have that \( h \) is not an element of \( H \).

Now assume \( f \) satisfies the conditions in (1). If \( \text{shift}(f_p) = |X| \), then since \( f_p \in C_G(f) \), we may take \( h = f_p \). Thus assume that \( \text{shift}(f_p) < |X| \), and so \( |X| = |X - S(f_p)| = |X - S(f)| + |S(f_a)| \). Suppose first that \( |X - S(f)| = |X| \). Choose a permutation \( q \) of \( X - S(f) \) that moves every point of \( X - S(f) \), and let \( h \in G_X \) coincide with \( q \) on \( X - S(f) \) and be the identity otherwise. Then \( h \in C_G(f) \) with \( \text{shift}(h) = |X| \), as required.

We may assume now that \( |X - S(f)| < |X| \), so that \( |X - S(f_a)| = |X - S(f)| + \text{shift}(f_p) < |X| \). Hence by (1b) above, we have that \( \text{def}(f) \geq 2 \), and also \( \text{shift}(f_e) = |X| \). Let \( B \) be the set of all the chains in \( f_e \), and recall that \( |B| = \text{def}(f) \). Take an index set \( I \) with \( |I| = |B| \) if \( B \) is infinite, and \( |I| = 1 \) if \( B \) is finite. Choose \( |I| \) disjoint doubleton subsets \( B_i \) of \( B \), where \( i \in I \), and let \( B_i = \{ q_i, r_i \} \), where \( q_i = (x_1x_2...), r_i = (y_1y_2...) \). For each \( i \in I \) choose a permutation \( t_i \) of \( X \) with \( S(t_i) = S(q_i) \cup S(r_i) \) that interchanges \( x_j \)'s and \( y_j \)'s; that is, for \( j = 1, 2, 3,... \) we have that \( t_i(x_j) = y_j, t_i(y_j) = x_j \) and \( t_i(x) = x \) for all \( x \in X - (S(q_i) \cup S(r_i)) \). Then by Proposition 3.5, each permutation \( t_i \in C_G(f) \). Observe that the permutations \( t_i \) are pairwise disjoint, and take \( h \) to be the (formal) product of all \( t_i \)'s where \( i \in I \). By Proposition 3.5 again, the permutation \( h \) is in \( C_G(f) \). Since for each \( i \in I \), \( \text{shift}(t_i) = \aleph_0 \), we have that \( \text{shift}(h) = \text{max}(|\aleph_0|, |I|) \). If \( |X| = \aleph_0 \), then \( \text{shift}(h) = |X| \). If \( |X| > \aleph_0 \), then since \( |X| = \text{shift}(f_e) = \aleph_0|B| \), we have that \( |B| = |X| \), so \( |I| = |B| = |X| \), and again \( \text{shift}(h) = |X| \), as required.

**Lemma 3.8.** Let \( Y \) be a subset of \( X \), and let \( q \) be a permutation of \( Y \) having no infinite cycles in its cyclic decomposition. Then \( C_{G_Y}(q) \cap S(Y, \aleph_0) \subseteq A_Y \) if and only if

1. \( |Y - S(q)| \leq 1 \), and
2. \( q \) is a product of disjoint cycles of distinct odd lengths.

**Proof:** Write \( q = \prod \{ \alpha_i : i \in I \} \) as a product of disjoint cycles \( \alpha_i \).

Suppose that \( C_{G_Y}(q) \cap S(Y, \aleph_0) \subseteq A_Y \). Then \( |Y - S(q)| \leq 1 \) (else any 2-cycle \( (xy) \) with \( x, y \in Y - S(q) \) is an odd permutation in \( C_{G_Y}(q) \)). To prove (2), recall that \( q = \prod \{ \alpha_i : i \in I \} \), and so for any finite subset \( J \) of \( I \), the permutation \( q_J = \prod \{ \alpha_i : i \in J \} \) is in \( C_{G_Y}(q) \). By our assumption \( q_J \) is an even permutation, therefore each \( \alpha_i, i \in I \),
has an odd length. If $\alpha_i$ and $\alpha_j$ are two distinct cycles in $q$ of the same odd length, $\alpha_i = (x_1x_2\ldots x_k) \neq (y_1y_2\ldots y_k) = \alpha_j$, then $t = \Pi \{ (x_m,y_m) : m = 1,2,\ldots, k \}$ is an odd finite permutation in $C_{G_Y}(q)$, a contradiction. Therefore $|\alpha_i| \neq |\alpha_j|$ if $i \neq j$.

Conversely, if $q$ is a product of disjoint cycles $\alpha_i$ of odd distinct length, for $i \in I$, then the group $\langle \alpha_i : i \in I \rangle$, is a subgroup of $C_{G_Y}(q) \cap A_Y$. Assume that $|Y - S(q)| \leq 1$. We show that, in fact, $\langle \alpha_i : i \in I \rangle = C_{G_Y}(q) \cap S(Y, N_0)$. Indeed, let $h \in C_{G_Y}(q) \cap S(Y, N_0)$ and let

$$Z = \{ q^k(x) : x \in S(h), k \text{ is an integer} \}.$$ 

Since shift($h$) is finite and $q$ is a product of finite cycles, the set $Z$ is finite. Moreover, if $\alpha = (x_1x_2\ldots x_m)$ is a cycle in $q$ such that $x_i \in S(h)$, for some $i = 1,2,\ldots, m$, then by the definition of $Z$, the set $\{x_1, x_2,\ldots, x_m\}$ is a subset of $Z$. If $|X - S(q)| = 1$, then $\{y\} = X - S(q)$ is not $Z$, since $h$ has to map the single one-cycle $(y)$ onto itself (Observation 3.3). Therefore the restriction $q|Z$ of $q$ to $Z$ is a permutation of $Z$ that moves every element of $Z$. Without loss of generality assume that $\alpha_1|Z, \alpha_2|Z,\ldots, \alpha_n|Z$ are the restrictions of cycles in $q$ that move the points of $Z$, and note that $S(\alpha_i) = S(\alpha_i|Z)$, for $i = 1,2,\ldots, m$. Write $q|Z = \alpha_1|Z\alpha_2|Z\ldots \alpha_n|Z$. Since $S(h) \subseteq Z$, we have that $h|Z \in C_{G_Z}(q|Z)$.

Set $T = \langle \alpha_1|Z, \alpha_2|Z,\ldots, \alpha_n|Z \rangle$, and let $\alpha_i|Z$ be an $m_i$-cycle, $m_i \geq 3$. Then $T$ is a subgroup of $C_{G_Z}(q|Z)$ of size $|T| = m_1m_2\ldots m_n$. The number of elements in $C_{G_Z}(q|Z)$ equals $|G_Z|$ divided by the number of conjugates of $q|Z$ in $G_Z$. Since the number of conjugates of $q|Z$ in $G_Z$ equals the number $|Z|/(m_1!m_2!\ldots m_n!)$ of partitions of $Z$ into classes of sizes $m_1, m_2,\ldots, m_n$, multiplied by the number $(m_1-1)!(m_2-1)\ldots(m_n-1)!$ of distinct $m_i$-cycles on the elements of the $m_i$-class, we see that in fact $T = C_{G_Z}(q|Z)$ and $h|Z \in T$. Therefore $h \in \langle \alpha_i : i \in I \rangle$.

4. $H$-NORMAL SEMIGROUPS

In this section we characterise those pairs $(H, f)$ of normal subgroups $H$ of the symmetric group $G_X$ and one-to-one transformations $f$ of $X$, that produce $H$-normal semigroups $(f : H)$ (having the property that $H = G_{(f:H)}$).

**Lemma 4.1.** Let $f \in W_X$ be a transformation with a finite non-zero defect, and let $H \leq G_X$. Then $G_{(f:H)} \leq HC_{G_X}(f)$. If additionally $C_{G_X}(f) \leq N(H)$ then $G_{(f:H)} = HC_{G_X}(f)$.

**Proof:** Take $g \in G_{(f:H)}$, then $gfg^{-1} \in (f : H)$, and so there exist permutations $q_1, q_2,\ldots, q_m \in H$ such that

$$gfg^{-1} = q_1f q_1^{-1}q_2f q_2^{-1}\ldots q_mf q_m^{-1}.$$ 

Then by Lemma 2.2, $\text{def}(f) = \text{def}(gfg^{-1}) = \text{def}(q_1f q_1^{-1}) + \text{def}(q_2f q_2^{-1}) + \ldots + \text{def}(q_mf q_m^{-1}) = m \text{ def}(f)$, so that $m = 1$. Hence $gfg^{-1} = q_1f q_1^{-1}$, and so $q_1^{-1}g \in C_{G_X}(f)$. Therefore $G_{(f:H)} \leq HC_{G_X}(f)$. 


Now assume that \( C_{\mathcal{G}_X}(f) \leq N(H) \) and take \( h \in H \) and \( t \in C_{\mathcal{G}_X}(f) \). Then for any element \( q_1 f q_1^{-1} q_2 f q_2^{-1} \ldots q_m f q_m^{-1} \in \langle f : H \rangle \) its conjugate by \( ht \) is a product of the conjugates of \( f \) of the form \( ht q_i f q_i^{-1} t^{-1} h^{-1} = ht q_i^{-1} f t q_i^{-1} t^{-1} h^{-1} \in \langle f : H \rangle \) since \( t q_i^{-1} t^{-1} h^{-1} \in H \) for all \( i \). Therefore \( ht \in G_{(f:H)} \).

**THEOREM 4.2.** Let \( f \in \mathcal{W}_X - \mathcal{G}_X \) and write \( f \) as a product of disjoint transformations \( f = f_p f_c \), where \( f_p \in \mathcal{G}_X \), \( f_c \in \mathcal{C}_X \). Take \( H \leq \mathcal{G}_X \). Then \( G_{(f:H)} = H \) if and only if one of the following holds:

1. \( H = \mathcal{G}_X \),
2. \( H = S(X, \mathbb{N}_o), \ |X| = \mathbb{N}_o, \ \text{def}(f) = 1, \ |X - S(f_c)| < \mathbb{N}_o, \ |X - S(f)| \leq 1, \text{and } f_p \) is a product of disjoint cycles of distinct odd lengths,
3. \( H = \mathcal{A}_X, \ |X| = \mathbb{N}_o, \ \text{def}(f) = 1, \ |X - S(f_c)| < \mathbb{N}_o, \ |X - S(f)| \leq 1 \text{, and } f_p \) is a product of disjoint cycles of distinct odd lengths.
4. \( H = \{i_X\}, \ |X| = \mathbb{N}_o, \ \text{def}(f) = 1, \ |X - S(f_c)| \leq 1 \).

**PROOF:** Suppose that \( H \) is a proper normal subgroup of \( \mathcal{G}_X \), so that \( H \leq S(X, |X|) \), and assume that \( H = G_{(f:H)} \) for a one-to-one transformation \( f \). By Proposition 3.7, we have that \( \text{def}(f) = 1 \) and \( |X - S(f_c)| < |X| \) so that \( |S(f_c)| = |X| \). Since the defect of \( f \) is 1, \( f_c \) consists of a single chain, and so \( |S(f_c)| = \mathbb{N}_o \). Therefore \( X \) is countable and \( X - S(f_c) \) is at most finite. By Lemma 4.1 we have that \( H = G_{(f:H)} = \mathcal{H}_{\mathcal{G}_X}(f) \), so by Corollary 3.6,

\[
\{ h \in \mathcal{C}_{\mathcal{G}_X}(f_p) : h(x) = x \text{ for all } x \in S(f_c) \} = \mathcal{C}_{\mathcal{G}_X}(f) \leq H.
\]

When \( X \) is countable the only non-trivial proper normal subgroups of \( \mathcal{G}_X \) are \( S(X, \mathbb{N}_o) \) and \( \mathcal{A}_X \). If \( H = \mathcal{A}_X \), then it follows from Lemma 3.8 that \( f \) can fix at most one point of \( X \) and \( f_p \) is a product of disjoint cycles of distinct odd lengths. If \( H = \{i_X\} \) then \( \mathcal{C}_{\mathcal{G}_X}(f) = \{i_X\} \) so that \( \mathcal{C}_{\mathcal{G}_X}(f_p) = \{i_X\} \) and hence \( |X - S(f_c)| \leq 1 \).

For the converse note that \( H \leq G_{(f:H)} \leq \mathcal{G}_X \) for any subgroup \( H \) of \( \mathcal{G}_X \), therefore if \( H = \mathcal{G}_X \) we have that \( G_{(f:G_X)} = \mathcal{G}_X \). Now assume that \( H \leq S(X, \mathbb{N}_o) \), \( X \) is countable, \( \text{def}(f) = 1 \) and \( X - S(f_c) \) is finite. Then by Lemma 4.1 and Corollary 3.6, we have that \( G_{(f:H)} = \mathcal{H}_{\mathcal{G}_X}(f) = H \{ h \in \mathcal{C}_{\mathcal{G}_X}(f_p) : h(x) = x \text{ for all } x \in S(f_c) \} \). Since \( X - S(f_c) \) is finite, \( \mathcal{C}_{\mathcal{G}_X}(f) \leq S(X, \mathbb{N}_o) \), so \( G_{(f:H)} = H \) for \( H = S(X, \mathbb{N}_o) \).

If we assume additionally that \( f_p \) is a product of disjoint cycles of distinct odd lengths, and \( f \) fixes at most one point, then Corollary 3.6 and Lemma 3.8 imply that \( \mathcal{C}_{\mathcal{G}_X}(f) \leq \mathcal{A}_X \), and so \( G_{(f:A_X)} = \mathcal{A}_X \). Similarly, if \( |X - S(f_c)| \leq 1 \), then \( \mathcal{C}_{\mathcal{G}_X}(f) = \{i_X\} \), and \( G_{(f: \{i_X\})} = \{i_X\} \).

5. **AUTOMORPHISMS**

If \( S \) is a semigroup of total transformations of a finite set \( X \), and \( G_S \) contains the alternating group \( \mathcal{A}_X \) on \( X \), then \( G_S = \mathcal{G}_X \), \( S \) is a \( \mathcal{G}_X \)-normal semigroup, all the
automorphisms of $S$ are inner, and the automorphism group $\text{Aut}(S)$ of $S$ is isomorphic to $G_X$ [6]. For an infinite set $X$ the fact that $A_X \leq G_S$ does not imply that $G_S = G_X$. However it will be shown in this section that if $S \not\leq G_X$ is a semigroup of one-to-one transformations of an infinite set $X$ such that $G_S$ contains $A_X$, then $S$ has the inner automorphism property. The technique used here is based on that of [3] developed to describe the automorphisms of $G_X$-normal semigroups.

Everywhere in this section we assume that $S$ is a subsemigroup of $\mathcal{W}_X$ that contains transformations with non-zero defects, and that $A_X \leq G_S$. To describe the automorphism group $\text{Aut}(S)$, in view of Equation 1 (in Section 1), we need to know the structure of the centraliser of the semigroup $S$ in $G_X$.

**PROPOSITION 5.1.** The centraliser $C_{G_X}(S)$ of $S$ is equal to $\{i_X\}$.

**Proof:** Let $f \in S$ and let $T = \langle f : A_X \rangle$ be a subsemigroup of $S$. We show that $C_{G_X}(T) = \{i_X\}$, and deduce the statement of the Proposition from an observation that since $T$ is a subsemigroup of $S$, the centraliser $C_{G_X}(S) \subseteq C_{G_X}(T)$. First we demonstrate that

\[ C_{G_X}(T) = \cap \{hC_{G_X}(f)h^{-1} : h \in A_X\}. \]

Indeed for each $q \in C_{G_X}(T)$ and $h \in A_X$ we have that $qhfh^{-1}q^{-1} = hfh^{-1}$, so that $h^{-1}qh \in C_{G_X}(f)$, and $q \in hC_{G_X}(f)h^{-1}$. Conversely, assume that $p \in \cap \{hC_{G_X}(f)h^{-1} : h \in A_X\}$ and take $g = h_1fh_1^{-1}h_2fh_2^{-1}\ldots h_mfh_m^{-1} \in T$. For each $i = 1, 2, \ldots, m$, there exists $r_i \in C_{G_X}(f)$ such that $p = h_ir_ih_i^{-1}$. Therefore $ph_ifh_i^{-1}p^{-1} = h_ir_ih_i^{-1}h_ifh_i^{-1}h_ir_i^{-1}h_i = h_ifh_i^{-1}$, so that $h_ifh_i^{-1}$, and $g \in C_{G_X}(T)$.

Take $g \in C_{G_X}(T) \subseteq C_{G_X}(f)$, and suppose that $g$ maps a chain $(x_1x_2x_3\ldots)$ of $f$ to a different chain $(g(x_1)g(x_2)g(x_3)\ldots)$ of $f$ (Proposition 3.5). Take $s = (x_1x_2x_3) \in A_X$. By Equation (2) above, $g = sqsq^{-1}$ for some $q \in C_{G_X}(f)$, and this $q$ has to map every chain of $f$ onto a chain in $f$ in prescribed order (Proposition 3.5). However we have that $q(x_1) = s^{-1}gs(x_1) = s^{-1}g(x_2) = g(x_2) = x_3$, hence $q(x_1x_2x_3\ldots)$ is not a chain in $f$. This contradiction proves that $g$ fixes every point of $S(f_c)$.

Suppose now that there is an $x \in X - S(f_c)$ such that $g(x) = y \neq x$, and note that $y \in X - S(f_c)$. Choose $z \in S(f_c)$ and take $s_1 = (xyz) \in A_X$. By Equation (2) again, $g = s_1qs_1^{-1}$ for some $q_1 \in C_{G_X}(f)$. However, in this case $q_1(z) = s_1^{-1}gs_1(z) = s_1^{-1}g(x) = s_1^{-1}(y) = x$, so $q_1(S(f_c)) \neq S(f_c)$, a contradiction to the fact that $q_1 \in C_{G_X}(f)$ (Proposition 3.5 again). Therefore $g$ is the identity permutation of $X$.

We proceed with the description of $\text{Aut}(S)$. For an $x \in X$ define

$\mathcal{R}_x = \{r \in S : x \in X - \text{im}(r)\}$.

In as much as $G_S$ contains a transitive group $A_X$, the set $\mathcal{R}_x$ is non-empty for every $x \in X$. In fact $\mathcal{R}_x$ is a right ideal of $S$, termed a point right ideal. Moreover, for
any distinct points \(x, y \in X\), the corresponding point right ideals \(R_x\) and \(R_y\) are also distinct. Indeed if \(r \in R_x \cap R_y\), choose distinct points \(u, v \in \text{im}(r)\) and take \(h = (yuv)\) to be a three-cycle in \(A_X \leq G_S\). Then \(\text{im}(hrh^{-1}) = h(\text{im}(r))\), and so \(hrh^{-1} \in R_x - R_y\). Therefore there is a one-to-one correspondence between the points \(x\) of \(X\) and the point right ideals \(R_x\) of \(S\).

We show that any automorphism of \(S\) acts faithfully on the set \(\{R_x : x \in X\}\) of all the point right ideals of \(S\). Given distinct transformations \(s\) and \(t\) in \(S\), define
\[
R(s, t) = \{r \in S : sr = tr\}.
\]

If non-empty, \(R(s, t)\) is a right ideal of \(S\) termed a function right ideal. It is not difficult to see that there is a relationship between non-empty function right ideals and point right ideals of \(S\) (see [3]) given by

\[
R(s, t) = \cap \{R_x : s(x) \neq t(x)\}.
\]

**Lemma 5.2.** For each \(x \in X\) there exist transformations \(s, t \in S\) such that \(R_x = R(s, t)\).

**Proof:** Since the defect of a product of two one-to-one transformations is the sum of their defects, and since \(S\) contains transformations with non-zero defects, we may choose a transformation \(g \in S\) with \(\text{def}(g) \geq 3\). Since \(G_S\) contains a transitive group \(A_X\) we may assume without loss of generality that \(x \in X - \text{im}(g)\). Let \(g(x) = y\), and choose two other distinct points \(u\) and \(z\) in \(X - \text{im}(g)\). Take three-cycles \(h_1 = (xzu)\) and \(h_2 = (xzy)\) in \(A_X \leq G_S\), and let \(s = h_1gh_1^{-1}g\) and \(t = h_2gh_2^{-1}g\).

We show that the above defined \(s\) and \(t\) are the required transformations. Indeed, \(s(x) = h_1gh_1^{-1}g(x) = h_1gh_1^{-1}(y) = h_1g(y) = g(y)\), since \(g(y)\) is not an element of \(\{x, u, z\} \subseteq X - \text{im}(f)\). Also \(t(x) = h_2gh_2^{-1}g(x) = h_2gh_2^{-1}(y) = h_2g(z) = g(z)\), since \(g(z) \neq g(x) = y\), and \(g(z) \neq x, z \in X - \text{im}(g)\), therefore \(s(x) \neq t(x)\). If \(a \neq x\), then \(g(a) \notin \{x, y, u, z\}\), so \(h_1^{-1}g(a) = h_2^{-1}g(a) \notin \{x, y, u, z\}\), and it is easy to see that \(s(a) = t(a)\).

The set of function right ideals is partially ordered by set inclusion, and its maximal elements are of the form \(R(s, t)\) where \(s\) and \(t\) differ precisely on one point of \(X\) (Equation 3 and Lemma 5.2). Formally:

**Lemma 5.3.** Given transformations \(s, t \in S\), \(R(s, t)\) is a maximal function right ideal of \(S\) if and only if \(R(s, t) = R_x\), for some \(x \in X\).

Take an automorphism \(\varphi\) of \(S\) and observe that \(\varphi\) acts on the set of function right ideals:
\[
\varphi(R(s, t)) = \{\varphi(r) : r \in S, \varphi(sr) = \varphi(tr)\}
= \{r' : r' \in S, \varphi(s)r' = \varphi(t)r'\}
= R(\varphi(s), \varphi(t)).
\]
Moreover $\varphi$ maps the set of all maximal function right ideals onto itself, hereby giving rise to a permutation $h$ of $X$ such that for an $x \in X$, $h(x) = y$ if $\varphi(\mathcal{R}_x) = \mathcal{R}_y$ (Lemma 5.3). The next result follows then from the observation that for any $x \in X$ and $f \in S$ we have that $x \in X - \text{im}(f)$ if and only if $f \in \mathcal{R}_x$ if and only if $\varphi(f) \in \varphi(\mathcal{R}_x) = \mathcal{R}_{h(x)}$.

**Lemma 5.4.** Given $f \in S$, $\text{im}(\varphi(f)) = h(\text{im}(f))$.

To see that $\varphi$ indeed acts on $S$ by conjugation by $h$, take an arbitrary $x \in X$, $f \in S$, and choose a non-permutation $g$ in $S$ with $x \in \text{im}(g)$. Take $u \in \text{im}(g)$ with $u \neq x$ and $v \in X - \text{im}(g)$, and let $q = (uxv) \in \mathcal{A}_X \subseteq G_S$. Then $qq^{-1} \in S$ and $\text{im}(qq^{-1}) = q(\text{im}(g)) = \text{im}(g) - \{x\} \cup \{v\}$, so that $\text{im}(g) - \text{im}(qq^{-1}) = \{x\}$. By Lemma 5.4,

$$\varphi(f)(h(x)) = \varphi(f)(\varphi(g)) - \text{im}(\varphi(qq^{-1}))$$

$$= \text{im}(\varphi(fg)) - \text{im}(\varphi(fqq^{-1}))$$

$$= hf(x),$$

and so $\varphi(f) = hfh^{-1}$. The above discussion together with Proposition 5.1 implies the next result.

**Theorem 5.5.** Let $X$ be an infinite set, and let $S$ be a semigroup of one-to-one transformations of $X$ that contains non-permutations. If the alternating group $\mathcal{A}_X$ is a subgroup of $G_S$, then each automorphism $\varphi$ of $S$ is inner, and $\text{Aut}(S) \cong G_S$.

**Corollary 5.6.** Let $f \in \mathcal{W}_X$ be a transformation with a non-zero defect, and let $H$ be a normal subgroup of $G_S$, then

1. $\text{Aut}(\langle f : H \rangle) = \text{Inn}(\langle f : H \rangle)$,
2. if $H \neq \{i_X\}$ and $f$ has a finite defect, then $\text{Aut}(\langle f : H \rangle) = \text{Inn}(\langle f : H \rangle) \cong HC_{G_X}(f)$.

**Proof:** To prove the first part of the Corollary, note that if $H$ is a non-trivial normal subgroup of $G_X$, then the result follows from Theorem 5.5. If $H = \{i_X\}$, then $\langle f : H \rangle$ is the monogenic semigroup generated by $f$. Since $f \in \mathcal{W}_X - G_X$, for any integer $k \geq 2$ we have that $f^k \neq f$ and so the identity automorphism is the only automorphism of $\langle f : H \rangle$.

The second part of the Corollary follows directly from Theorem 5.5 and Lemma 4.1.

Observe that if $H$ is a proper normal subgroup of $G_X$ and $f \in \mathcal{W}_X$ is a non-permutation satisfying $\text{Aut}(\langle f : H \rangle) \cong H$, then by Proposition 3.7 and Corollary 5.6, we have that $\text{def}(f) = 1$ and $|X - S(f_0)| < |X|$, so that $X$ is a countable set.

**Corollary 5.7.** Let $X$ be a countable set. Then there exists a non-permutation $f \in \mathcal{W}_X$ such that for any normal subgroup $H$ of $G_X$ we have that $\text{Aut}(\langle f : H \rangle) \cong H$. 


PROOF: Take $f$ to be a single chain shifting all the points of $X$. Then, by Corollary 3.6, $C_{G_X}(f) = \{i_X\}$. The result follows from Corollary 5.6.

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