Gaussian integer Straus-Erdös

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In this paper we are attempting to prove the conjecture made in our previous paper [**3**]. This conjecture is given as follows:

CONJECTURE 0.1. *Let $a+bi \in \mathbb{Z}[i]$ such that both $a \geq 0$ and $b \geq 0$ with $a+bi \notin \{0,1,i,1+i\}$. There exists $x_1+y_1i, x_2+y_2i, x_3+y_3i \in \mathbb{Z}[i]$ such that $x_1y_1 \geq 0$, $x_2y_2 \geq 0$, $x_3y_3 \geq 0$ and*

$$\frac{4}{a+bi} = \frac{1}{x_1+y_1i} + \frac{1}{x_2+y_2i} + \frac{1}{x_3+y_3i}.$$

To prove this we first divide $a+bi$ by 4 and look at the possible remainders:

$$\begin{array}{cccc} 0 & i & 2i & -i \\ 1 & 1+i & 1+2i & 1-i \\ 2 & 2+i & 2+2i & 2-i \\ -1 & -1+i & -1+2i & -1-i. \end{array}$$

To prove the conjecture we must find a decomposition in each of these cases. We start with the easiest cases and work our way up to the harder cases.

PROPOSITION 0.2. *Suppose that for $a+bi \in \mathbb{Z}[i]$ there exists $x_1+y_1i, x_2+y_2i, x_3+y_3i \in \mathbb{Z}[i]$ such that $x_1y_1 \geq 0$, $x_2y_2 \geq 0$, $x_3y_3 \geq 0$ and*

$$\frac{4}{a+bi} = \frac{1}{x_1+y_1i} + \frac{1}{x_2+y_2i} + \frac{1}{x_3+y_3i}.$$

We see that the same will be true for $b+ai$ because

$$\frac{4}{b+ai} = \frac{1}{y_1+x_1i} + \frac{1}{y_2+x_2i} + \frac{1}{y_3+x_3i}.$$

Proof.
By supposition we have that for $a+bi \in \mathbb{Z}[i]$ there exists $x_1+y_1i, x_2+y_2i, x_3+y_3i \in \mathbb{Z}[i]$ such that $x_1y_1 \geq 0$, $x_2y_2 \geq 0$, $x_3y_3 \geq 0$ and

$$\frac{4}{a+bi} = \frac{1}{x_1+y_1i} + \frac{1}{x_2+y_2i} + \frac{1}{x_3+y_3i}.$$

Note that for $a+bi \in \mathbb{Z}[i]$ we have that

$$\frac{1}{\overline{(a+bi)}} = \overline{\left(\frac{1}{a+bi}\right)}.$$

$$\begin{aligned}
\frac{4}{b+ai} &= \frac{4}{i\cdot(a-bi)} \\
&= \frac{4}{i\cdot\overline{(a+bi)}} \\
&= \frac{1}{i}\cdot\overline{\left(\frac{4}{a+bi}\right)} \\
&= \frac{1}{i}\cdot\overline{\left(\frac{1}{x_1+y_1 i}+\frac{1}{x_2+y_2 i}+\frac{1}{x_3+y_3 i}\right)} \\
&= \frac{1}{i}\cdot\left(\frac{1}{\overline{x_1+y_1 i}}+\frac{1}{\overline{x_2+y_2 i}}+\frac{1}{\overline{x_3+y_3 i}}\right) \\
&= \frac{1}{i\cdot(x_1-y_1 i)}+\frac{1}{i\cdot(x_2-y_2 i)}+\frac{1}{i\cdot(x_3-y_3 i)} \\
&= \frac{1}{y_1+x_1 i}+\frac{1}{y_2+x_2 i}+\frac{1}{y_3+x_3 i}.
\end{aligned}$$

■

It suffices to find decompositions for $4/(a+bi)$ where we can assume that $a \geq b$ and $a, b \geq 0$. We will break this into four classes of related decompositions.

1. REMAINDER MULTIPLES OF 2

In this section we prove the four easiest class of cases. The proofs are trivial because of the nature of the the remainder.

Remainder 0

Assume that $a, b \geq 0$ and $a+bi \neq 0$.

$$\begin{aligned}
\frac{4}{4(a+bi)+0} &= \frac{1}{a+bi} \\
&= \frac{1}{(3a)+(3b)i}+\frac{1}{(3a)+(3b)i}+\frac{1}{(3a)+(3b)i}.
\end{aligned}$$

Remainder 2

Assume that $a, b \geq 0$.

$$\begin{aligned}
\frac{4}{4(a+bi)+2} &= \frac{2}{2(a+bi)+1} \\
&= \frac{1}{(2a+1)+(2b)i}+\frac{1}{(4a+2)+(4b)i}+\frac{1}{(4a+2)+(4b)i}.
\end{aligned}$$

Remainder $2i$
Assume that $a, b \geq 0$.

$$\frac{4}{4(a+bi)+2i} = \frac{2}{2(a+bi)+i}$$
$$= \frac{1}{(2a)+(2b+1)i} + \frac{1}{(4a)+(4b+2)i} + \frac{1}{(4a)+(4b+2)i}.$$

Remainder $2+2i$
Assume that $a, b \geq 0$.

$$\frac{4}{4(a+bi)+(2+2i)} = \frac{2}{2(a+bi)+(1+i)}$$
$$= \frac{1}{(2a+1)+(2b+1)i} + \frac{1}{(4a+2)+(4b+2)i} + \frac{1}{(4a+2)+(4b+2)i}.$$

2. REMAINDER ASSOCIATES OF $1+I$

In this section we address the class that I consider to be the next easiest to decompose. This means that a basic algorithm satisfies a vast majority of this class. We begin with the two scenarios for which this algorithm completely takes care of the remainder cases.

Remainder $1+i$
Assume that $a, b \geq 0$ and $a+bi \neq 0$. Recall that it suffices to consider $a \geq b$. We first assume that $2(a^2-b^2)-4ab-b \leq 0$. We see that

$$\frac{4}{4(a+bi)+(1+i)} = \frac{1}{a+bi} - \frac{1+i}{(a+bi)(4(a+bi)+(1+i))}$$
$$= \frac{1}{a+bi} - \frac{1}{(a+bi)((2-2i)(a+bi)+1)}$$
$$= \frac{1}{a+bi} - \frac{1}{(2a^2+4ab-2b^2+a)+(2b^2+4ab-2a^2+b)i}$$

Because $a \geq b \geq 0$ we see that $2a^2+4ab-2b^2+a \geq 0$. Because $2(a^2-b^2)-4ab-b \leq 0$ we see that $2b^2+4ab-2a^2+b \geq 0$.

Next assume that $2(a^2-b^2)-4ab-b \geq 0$ and $a+b$ is odd. We see that

$$\begin{aligned}
\frac{4}{4(a+bi)+(1+i)} &= \frac{1}{(a+1)+bi} + \frac{3-i}{((a+1)+bi)(4(a+bi)+(1+i))} \\
&= \frac{1}{(a+1)+bi} \\
&\quad + \frac{3-i}{(4(a^2-b^2)+4a+(a-b+1))+(8ab+4b+(a+b+1))i} \\
&= \frac{1}{(a+1)+bi} \\
&\quad + \frac{2}{(4(a^2-b^2)+4a+(a-b+1))+(8ab+4b+(a+b+1))i} \\
&\quad + \frac{1-i}{(4(a^2-b^2)+4a+(a-b+1))+(8ab+4b+(a+b+1))i}
\end{aligned}$$

Because $a+b$ is odd, both $a-b+1$ and $a+b+1$ are even. This means that for the second fraction, 2 can be factored out of the real and imaginary part of the denominator. The second fraction is unit. Naturally $8ab+4b+(a+b+1) \geq 0$. Because $a > b$, we see that $4(a^2-b^2)+4a+(a-b+1) \geq 0$.

Multiplying the third fraction in the numerator and denominator by $1+i$ we see that it is equivalent to

$$\frac{2}{(4(a^2-b^2)-8ab+4(a-b)-2b)+(4(a^2-b^2)+8ab+4(a+b)+2a+2)i}.$$

Canceling out a factor of 2 makes this fraction

$$\frac{1}{(2(a^2-b^2)-4ab+2(a-b)-b)+(2(a^2-b^2)+4ab+2(a+b)+a+1)i}.$$

Because both $4(a^2-b^2)+4a+(a-b+1) \geq 0$ and $8ab+4b+(a+b+1) \geq 0$, we see that their average $2(a^2-b^2)+4ab+2(a+b)+a+1 \geq 0$. Because $2(a^2-b^2)-4ab-b \geq 0$ and $a \geq b$ we see that $2(a^2-b^2)-4ab+2(a-b)-b \geq 0$.

Finally assume that $2(a^2-b^2)-4ab-b \geq 0$ and $a+b$ is even. We see that

$$\begin{aligned}
\frac{4}{4(a+bi)+(1+i)} &= \frac{1}{a+bi} - \frac{1+i}{(a+bi)(4(a+bi)+(1+i))} \\
&= \frac{1}{a+bi} - \frac{1+i}{(4(a^2-b^2)+(a-b))+(8ab+(a+b))i} \\
&= \frac{1}{a+bi} - \frac{2}{(4(a^2-b^2)+(a-b))+(8ab+(a+b))i} \\
&\qquad + \frac{1-i}{(4(a^2-b^2)+(a-b))+(8ab+(a+b))i}
\end{aligned}$$

Because $a+b$ is even, we see that $a-b$ is even. This means that 2 can be factored out of both the real and imaginary parts of the denominator of the second fraction. The second fraction is unit. Furthermore, notice that $a > b \geq 0$ implies that both $4(a^2-b^2)+(a-b) \geq 0$ and $8ab+(a+b) \geq 0$.

Next multiply the third fraction in the numerator and denominator by $1+i$. This will make the third fraction equivalent to

$$\frac{2}{(4(a^2-b^2)-8ab-2b)+(4(a^2-b^2)+8ab+2a)i}.$$

Canceling out a factor of 2 makes this fraction

$$\frac{1}{(2(a^2-b^2)-4ab-b)+(2(a^2-b^2)+4ab+a)i}.$$

Because $a \geq b \geq 0$ we see that $2(a^2-b^2)+4ab+a \geq 0$. Because $2(a^2-b^2)-4ab-b \geq 0$ by assumption both parts are positive.

Remainder $-(1-i)$
Assume that $a \geq 1$ and $b \geq 0$. Recall that it suffices to consider $a \geq b$. We first assume that $2(a^2-b^2)-4ab-a \geq 0$. We see that

$$\begin{aligned}\frac{4}{4(a+bi)-(1-i)} &= \frac{1}{a+bi}+\frac{1-i}{(a+bi)(4(a+bi)-(1-i))}\\ &= \frac{1}{a+bi}+\frac{1}{(a+bi)((2+2i)(a+bi)-1)}\\ &= \frac{1}{a+bi}+\frac{1}{(2a^2-4ab-2b^2-a)+(2a^2+4ab-2b^2-b)i}\end{aligned}$$

Because $a \geq 1$, $b \geq 0$ and $a \geq b$ we see that $2a^2+4ab-2b^2-b = 2(a^2-b^2)+b(4a-1) \geq 0$. We have that $2a^2-4ab-2b^2-a \geq 0$ by assumption.

Next assume that $2(a^2-b^2)-4ab-a \leq 0$ and $a+b$ is odd. We see that

$$\begin{aligned}\frac{4}{4(a+bi)-(1-i)} &= \frac{1}{(a-1)+bi}-\frac{3+i}{((a-1)+bi)(4(a+bi)-(1-i))}\\ &= \frac{1}{(a-1)+bi}\\ &\quad -\frac{3+i}{(4(a^2-b^2)-4a-(a+b-1))+(8ab-4b+(a-b-1))i}\\ &= \frac{1}{(a-1)+bi}\end{aligned}$$

$$- \frac{2}{(4(a^2-b^2)-4a-(a+b-1))+(8ab-4b+(a-b-1))i}$$
$$- \frac{1+i}{(4(a^2-b^2)-4a-(a+b-1))+(8ab-4b+(a-b-1))i}$$

Because $a+b$ is odd, both $a-b-1$ and $a+b-1$ are even. This means that for the second fraction, 2 can be factored out of the real and imaginary part of the denominator. The second fraction is unit. Because $2(a^2-b^2)-4ab-a \leq 0$ we have that $b \geq 1$ and recall that $a \geq b$ by supposition. This will imply that $4a(2b-1)-1 \geq 3$. We see then that $8ab-4a+(a-b-1) = 4a(2b-1)-1+(a-b) \geq 0$. Notice that $a \neq b$ because $a+b$ is odd. If we let $n \geq 1$ be an odd number so that $a = b+n$ we see that $4(a^2-b^2) = 8nb+4n^2$, $4a = 4b+4n$ and $a+b-1 = 2b+n-1$. This will imply that $4(a^2-b^2)-4a-(a+b-1) = (8nb-6b)+(4n^2-5n+1)$. Because $n \geq 1$ we see that $8nb-6b \geq 0$. Because $4n^2-5n+1 = (4n-1)(n-1)$ we see that both $4n-1 \geq 0$ and $n-1 \geq 0$. This will imply that $4(a^2-b^2)-4a-(a+b-1) \geq 0$.

Multiplying the third fraction in the numerator and denominator by $1-i$ we see that it is equivalent to

$$\frac{2}{(4(a^2-b^2)+8ab-4(a+b)-2b)+(-4(a^2-b^2)+8ab+4(a-b)+2a-2)i}.$$

Canceling out a factor of 2 makes this fraction

$$\frac{1}{(2(a^2-b^2)+4ab-2(a+b)-b)+(-2(a^2-b^2)+4ab+2(a-b)+a-1)i}.$$

Because both $4(a^2-b^2)-4a-(a+b-1) \geq 0$ and $8ab-4b+(a-b-1) \geq 0$, we see that their average $2(a^2-b^2)+4ab-2(a+b)-b \geq 0$. Because $2(a^2-b^2)-4ab-a \leq 0$ we see that $-2(a^2-b^2)+4ab+a \geq 0$. Because $a \geq b+1$ we see that $2(a-b)-1 \geq 0$. This will imply that $-2(a^2-b^2)+4ab+2(a-b)+a-1 \geq 0$.

Next we are going to assume that $a = b$. This will make $2(a^2-b^2)-4ab-a \leq 0$ and $a+b$ even. We see that

$$\begin{aligned}
\frac{4}{4(a+ai)-(1-i)} &= \frac{1}{a+ai} + \frac{1-i}{(a+ai)(4(a+ai)-(1-i))} \\
&= \frac{1}{a+ai} + \frac{1-i}{(-2a)+(8a^2)i} \\
&= \frac{1}{a+ai} - \frac{2i}{(-2a)+(8a^2)i} + \frac{1+i}{(-2a)+(8a^2)i} \\
&= \frac{1}{a+ai} - \frac{1}{(4a^2)+ai} + \frac{1}{(4a^2-a)+(4a^2+a)i}
\end{aligned}$$

Because $a \geq 1$ we have that all three fractions have positive values in the real and imaginary parts of the denominators.

Finally assume that $a \neq b$, $2(a^2 - b^2) - 4ab - a \leq 0$ and $a + b$ is even. We see that

$$\begin{aligned}\frac{4}{4(a+bi)-(1-i)} &= \frac{1}{a+bi} + \frac{1-i}{(a+bi)(4(a+bi)-(1-i))}\\ &= \frac{1}{a+bi} + \frac{1-i}{(4(a^2-b^2)-(a+b))+(8ab+(a-b))i}\\ &= \frac{1}{a+bi} + \frac{2}{(4(a^2-b^2)-(a+b))+(8ab+(a-b))i}\\ &\qquad - \frac{1+i}{(4(a^2-b^2)-(a+b))+(8ab+(a-b))i}\end{aligned}$$

Because $a + b$ is even, we see that $a - b$ is even. This means that 2 can be factored out of both the real and imaginary parts of the denominator of the second fraction. The second fraction is unit. Suppose $a = b + n$ where $n \geq 2$ is an even number. We see that $4(a^2 - b^2) - (a + b) = 8nb + 4n^2 - 2b - n = b(8n - 2) + n(4n - 1)$. Because $b(8n - 2) \geq 0$ and $n(4n - 1) \geq 0$ we see that $4(a^2 - b^2) - (a + b) \geq 0$. Furthermore, notice that $a \geq b$ implies that both $8ab + (a - b) \geq 0$.

Next multiply the third fraction in the numerator and denominator by $1 - i$. This will make the third fraction equivalent to

$$\frac{2}{(4(a^2-b^2)+8ab-2b)+(-4(a^2-b^2)+8ab+2a)i}.$$

Canceling out a factor of 2 makes this fraction

$$\frac{1}{(2(a^2-b^2)+4ab-b)+(-2(a^2-b^2)+4ab+a)i}.$$

Because $a \geq b$ and $a \geq 1$ we see that $2(a^2 - b^2) + 4ab - b = 2(a^2 - b^2) + b(4a - 1) \geq 0$. Because $2(a^2 - b^2) - 4ab - a \leq 0$ by assumption we see that $-2(a^2 - b^2) + 4ab + a \geq 0$.

For the next two remainder scenarios the algorithm takes care of almost all possibilities, but there are going to be an infinite number values in these cases unaccounted for. If we think of fitting carpet to a floor (the floor being the complex plane), the previous two cases had the carpet line up at a seam, while for the next two cases the carpet leaves a small gap along the seam that needs to be addressed.

Remainder $-(1 + i)$
Assume that $a, b \geq 1$. Recall that it suffices to consider $a \geq b$. We first assume that $2(a^2 - b^2) - 4ab + b \leq 0$. We see that

$$\begin{aligned}
\frac{4}{4(a+bi)-(1+i)} &= \frac{1}{a+bi} + \frac{1+i}{(a+bi)(4(a+bi)-(1+i))} \\
&= \frac{1}{a+bi} + \frac{1}{(a+bi)((2-2i)(a+bi)-1)} \\
&= \frac{1}{a+bi} + \frac{1}{(2a^2+4ab-2b^2-a)+(2b^2+4ab-2a^2-b)i}
\end{aligned}$$

Because $a \geq b \geq 1$ we see that $a^2-b^2 \geq 0$ and $a(4b-1) \geq 0$. This means that $2a^2+4ab-2b^2-a \geq 0 = 2(a^2-b^2)+a(4b-1) \geq 0$. Because $2(a^2-b^2)-4ab+b \leq 0$ we see that $2b^2+4ab-2a^2-b \geq 0$.

Next assume that $2(a^2-b^2)-4ab+b > 0$ and $a+b$ is even. We see that

$$\begin{aligned}
\frac{4}{4(a+bi)-(1+i)} &= \frac{1}{a+bi} + \frac{1+i}{(a+bi)(4(a+bi)-(1+i))} \\
&= \frac{1}{a+bi} + \frac{1+i}{(4(a^2-b^2)-(a-b))+(8ab-(a+b))i} \\
&= \frac{1}{a+bi} + \frac{2}{(4(a^2-b^2)-(a-b))+(8ab-(a+b))i} \\
&\qquad - \frac{1-i}{(4(a^2-b^2)-(a-b))+(8ab-(a+b))i}
\end{aligned}$$

Because $a+b$ is even, we see that $a-b$ is even. This means that 2 can be factored out of both the real and imaginary parts of the denominator of the second fraction. The second fraction is unit. Furthermore, notice that $a \geq b \geq 1$ implies that both $4(a^2-b^2)-(a-b) = (a-b)(4(a+b)-1) \geq 4(a+b)-1 \geq 0$. We also see that if $a = b+n$ where $n \geq 2$ is even, $8ab-(a+b) = b(8b-2)+n(8b-1) \geq 0$.

Next multiply the third fraction in the numerator and denominator by $1+i$. This will make the third fraction equivalent to

$$\frac{2}{(4(a^2-b^2)-8ab+2b)+(4(a^2-b^2)+8ab-2a)i}.$$

Canceling out a factor of 2 makes this fraction

$$\frac{1}{(2(a^2-b^2)-4ab+b)+(2(a^2-b^2)+4ab-a)i}.$$

Because $a \geq b \geq 1$ we see that $2(a^2-b^2)+4ab-a = 2(a^2-b^2)+a(4b-1) \geq 0$. Because $2(a^2-b^2)-4ab+b > 0$ by assumption both parts are positive.

Next assume that $2(a^2-b^2)-4ab+b-2(a-b) \geq 0$ and $a+b$ is odd. This necessarily means that $2(a^2-b^2)-4ab+b > 0$. We see that

$$\begin{aligned}\frac{4}{4(a+bi)-(1+i)} &= \frac{1}{(a-1)+bi} - \frac{3-i}{((a-1)+bi)(4(a+bi)-(1+i))} \\ &= \frac{1}{(a-1)+bi} \\ &\quad - \frac{3-i}{(4(a^2-b^2)-4a-(a-b-1))+(8ab-4b-(a+b-1))i} \\ &= \frac{1}{(a-1)+bi} \\ &\quad - \frac{2}{(4(a^2-b^2)-4a-(a-b-1))+(8ab-4b-(a+b-1))i} \\ &\quad - \frac{1-i}{(4(a^2-b^2)-4a-(a-b-1))+(8ab-4b-(a+b-1))i}\end{aligned}$$

Because $a+b$ is odd, both $a-b-1$ and $a+b-1$ are even. This means that for the second fraction, 2 can be factored out of the real and imaginary part of the denominator. The second fraction is unit. Because $2(a^2-b^2)-4ab+b>0$ we see that $a-b\geq 1$. This means that $4(a^2-b^2)-4a-(a-b-1)=(a-b)(4(a+b)-1)-4a+1\geq 4(a+b)-1-4a+1=4b\geq 0$. Recall that $b\geq 1$. We also see that if $a=b+n$ where $n\geq 1$ is odd, $8ab-4b-(a+b-1)=8b^2+8bn-6b-n+1$. We can express this as $b(8b-6)+n(8b-1)+1$ and we see that $b(8b-6)\geq 0$ and $n(8b-1)\geq 0$ so we see that $8ab-4b-(a+b-1)\geq 0$.

Multiplying the third fraction in the numerator and denominator by $1+i$ we see that it is equivalent to

$$\frac{2}{(4(a^2-b^2)-8ab-4(a-b)+2b)+(4(a^2-b^2)+8ab-4(a+b)-2a+2)i}.$$

Canceling out a factor of 2 makes this fraction

$$\frac{1}{(2(a^2-b^2)-4ab-2(a-b)+b)+(2(a^2-b^2)+4ab-2(a+b)-a+1)i}.$$

Because both $4(a^2-b^2)-4a-(a-b-1)\geq 0$ and $8ab-4b-(a+b-1)\geq 0$, we see that their average $2(a^2-b^2)+4ab-2(a+b)-a+1\geq 0$. By assumption we see that $2(a^2-b^2)-4ab+b-2(a-b)\geq 0$.

Finally assume that $2(a^2-b^2)-4ab+b-2(a-b)<0$, $2(a^2-b^2)-4ab+b>0$ and $a+b$ is odd. For this final case we can account for it by selecting the first two fractions very carefully.

Let the first fraction be

$$\frac{1}{a+bi}.$$

Let the second fraction be

$$\frac{1}{(2(a^2-b^2)+4ab-2(a+b)-a+1)+(-2(a^2-b^2)+4ab+2(a-b)-b)i}.$$

Notice that $2(a^2-b^2)+4ab-2(a+b)-a+1)+(-2(a^2-b^2 = 2(a+b)(a-b-1)+a(4b-1)+1)$. Because $a > b \geq 1$ we see that $2(a^2-b^2)+4ab-2(a+b)-a+1) \geq 0$. We see by supposition that $-2(a^2-b^2)+4ab+2(a-b)-b \geq 0$.

With this selection of the first two fractions we see that the third unit fraction must necessarily have a denominator real component $-2a^3-6a^2b+3a^2+4ab+6b^2a-a+2b^3-3b^2$ and imaginary component $2a^3-2a^2-6a^2b-6b^2a+6ab+2b^2-b+2b^3$.

Notice that the denominator can be written as $(4a-1)+(4b-1)i$ multiplied by

$$\frac{(-(a^2-b^2)-2ab+(a+b))+((a^2-b^2)-2ab-(a-b))i}{2}$$

We see that $2(a^2-b^2)-4ab+b-2(a-b) < 0$ implies that $(a^2-b^2)-2ab-(a-b) < 0$. We also see that $-(a^2-b^2)-2ab+(a+b) = (a+b)(1-(a-b))-2ab$. Because $a > b \geq 1$ we see that $-(a^2-b^2)-2ab+(a+b) < 0$. Because both $4a-1 \geq 0$ and $4b-1 \geq 0$ we necessarily have that $2a^3-2a^2-6a^2b-6b^2a+6ab+2b^2-b+2b^3 \leq 0$ through the product of the complex terms mentioned above.

Finally we see that $-2a^3-6a^2b+3a^2+4ab+6b^2a-a+2b^3-3b^2 = (-2(a+b)+3)(a^2-b^2)-4(a-b-1)ab-2a$. Because $a > b \geq 1$ we see that $-2(a+b)+3)(a^2-b^2) \leq 0$ and $-4(a-b-1)ab \leq 0$. This means that $-2a^3-6a^2b+3a^2+4ab+6b^2a-a+2b^3-3b^2 \leq 0$ and both the real and imaginary components of the denominator of this unit fraction are negative.

Remainder $1-i$

Assume that $a \geq b \geq 1$. We first assume that $2(a^2-b^2)-4ab+a \geq 0$. We see that

$$\begin{aligned}\frac{4}{4(a+bi)+(1-i)} &= \frac{1}{a+bi} - \frac{1-i}{(a+bi)(4(a+bi)+(1-i))}\\ &= \frac{1}{a+bi} - \frac{1}{(a+bi)((2+2i)(a+bi)+1)}\\ &= \frac{1}{a+bi} - \frac{1}{(2a^2-4ab-2b^2+a)+(2a^2+4ab-2b^2+b)i}\end{aligned}$$

Because $a \geq b \geq 0$ we see that $2a^2+4ab-2b^2+b \geq 0$. We have that $2a^2-4ab-2b^2+a \geq 0$ by assumption. This implies that both the real and imaginary parts of the denominator are positive.

Next assume that $2(a^2-b^2)-4ab+a<0$ and $a+b$ is even. We see that

$$\begin{aligned}\frac{4}{4(a+bi)+(1-i)} &= \frac{1}{a+bi} - \frac{1-i}{(a+bi)(4(a+bi)+(1-i))}\\ &= \frac{1}{a+bi} - \frac{1-i}{(4(a^2-b^2)+(a+b))+(8ab+(-a+b))i}\\ &= \frac{1}{a+bi} - \frac{2}{(4(a^2-b^2)+(a+b))+(8ab+(-a+b))i}\\ &\qquad + \frac{1+i}{(4(a^2-b^2)+(a+b))+(8ab+(-a+b))i}\end{aligned}$$

Because $a+b$ is even, we see that $a-b$ is even. This means that 2 can be factored out of both the real and imaginary parts of the denominator of the second fraction. The second fraction is unit. Furthermore, notice that $a \geq b \geq 1$ implies that both $4(a^2-b^2)+(a+b) \geq 0$ and $8ab+(-a+b) = a(8b-1)+b \geq 0$.

Next multiply the third fraction in the numerator and denominator by $1-i$. This will make the third fraction equivalent to

$$\frac{2}{(4(a^2-b^2)+8ab+2b)+(-4(a^2-b^2)+8ab-2a)i}.$$

Canceling out a factor of 2 makes this fraction

$$\frac{1}{(2(a^2-b^2)+4ab+b)+(-2(a^2-b^2)+4ab-a)i}.$$

Because $4(a^2-b^2)+(a+b) \geq 0$ and $8ab+(-a+b) \geq 0$ we see that their average $2(a^2-b^2)+4ab+b \geq 0$. By supposition we see that $2(a^2-b^2)-4ab+a<0$, so we have that $-2(a^2-b^2)+4ab-a>0$.

Next assume that $2(a^2-b^2)-4ab+a+2(a-b)+1 \leq 0$ and $a+b$ is odd. This necessarily implies that $2(a^2-b^2)-4ab+a<0$. We see that

$$\begin{aligned}\frac{4}{4(a+bi)+(1-i)} &= \frac{1}{(a+1)+bi} + \frac{3+i}{((a+1)+bi)(4(a+bi)+(1-i))}\\ &= \frac{1}{(a+1)+bi}\\ &\quad + \frac{3+i}{(4(a^2-b^2)+4a+(a+b+1))+(8ab+4b-(a-b+1))i}\\ &= \frac{1}{(a+1)+bi}\end{aligned}$$

$$+\frac{2}{(4(a^2-b^2)+4a+(a+b+1))+(8ab+4b-(a-b+1))i}$$
$$+\frac{1+i}{(4(a^2-b^2)+4a+(a+b+1))+(8ab+4b-(a-b+1))i}$$

Because $a+b$ is odd, both $a-b+1$ and $a+b+1$ are even. This means that for the second fraction, 2 can be factored out of the real and imaginary part of the denominator. The second fraction is unit. Because $a \geq b \geq 1$ we see that $4(a^2-b^2)+4a+(a+b+1) \geq 0$. Because $b \geq 1$ we see that $b-1 \geq 0, 4b \geq 0$ and $8b-1 \geq 0$. This will imply that $8ab+4b-(a-b+1) = a(8b-1)+4b+(b-1) \geq 0$.

Multiplying the third fraction in the numerator and denominator by $1-i$ we see that it is equivalent to

$$\frac{2}{(4(a^2-b^2)+8ab+4(a+b)+2b)+(-4(a^2-b^2)+8ab-4(a-b)-2a-2)i}.$$

Canceling out a factor of 2 makes this fraction

$$\frac{1}{(2(a^2-b^2)+4ab+2(a+b)+b)+(-2(a^2-b^2)+4ab-2(a-b)-a-1)i}.$$

Because both $4(a^2-b^2)+4a+(a+b+1) \geq 0$ and $8ab+4b-(a-b+1) \geq 0$, we see that their average $2(a^2-b^2)+4ab+2(a+b)+b \geq 0$. Because $2(a^2-b^2)-4ab+a+2(a-b)+1 \leq 0$ by supposition we see that $-2(a^2-b^2)+4ab-2(a-b)-a-1 \geq 0$. Both the real and imaginary parts are positive.

Finally assume that $2(a^2-b^2)-4ab+a+2(a-b)+1 > 0$, $2(a^2-b^2)-4ab+a < 0$ and $a+b$ is odd. For this final case we can account for it by selecting the first two fractions very carefully.

Let the first fraction be

$$\frac{1}{a+bi}.$$

Let the second fraction be

$$\frac{1}{(-2(a^2-b^2)+4ab-2(a+b)-a)+(-2(a^2-b^2)-4ab+2(a-b)-b+1)i}.$$

Because $a \geq b \geq 1$ it is clear that $-2(a^2-b^2)-4ab+2(a-b)-b+1 \leq 0$. By supposition we have that $-2(a^2-b^2)+4ab-2(a-b)-a-1 \leq 0$. Because $4b-1 \geq 0$, this will imply that $-2(a^2-b^2)+4ab-2(a+b)-a = -2(a^2-b^2)+4ab-2(a-b)-a-1-(4b-1) \leq 0$. This implies that both the real and imaginary part of the denominator of this unit fraction are negative.

With this selection of the first two fractions we see that the third unit fraction must necessarily have a denominator real component $2a^3+6a^2b-2a^2-6ab^2+6ab-a-$

$2b^3 + 2b^2$ and imaginary component $-2a^3 + 6a^2b - 3a^2 + 6ab^2 - 4ab - 2b^3 + 3b^2 - b$.

Notice that the denominator can be written as $(4a + 1) + (4b - 1)i$ multiplied by

$$\frac{((a^2 - b^2) + 2ab - (a - b)) + (-(a^2 - b^2) + 2ab - (a + b))i}{2}.$$

We see that $2(a^2 - b^2) - 4ab + 2(a - b) + a + 1 > 0$ implies that $-(a^2 - b^2) + 2ab - (a + b) < (1/2)((a + 1) - 4b)$. If we let $a = 4b + n$ where $n \geq 0$ we see that $2(a^2 - b^2) - 4ab + a = 14b^2 + 12bn + 4b + 2n^2 + n \geq 0$ which is a contradiction to our assumption that $2(a^2 - b^2) - 4ab + a < 0$. This implies that $(a + 1) - 4b \leq 0$ and therefore $-(a^2 - b^2) + 2ab - (a + b) < 0$. Because $a > b \geq 1$ we see that $(a^2 - b^2) + 2ab - (a - b) = (a - b)(a + b + 1) + 2ab > 0$. Because both $4a + 1 \geq 0$ and $4b - 1 \geq 0$ we necessarily have that $2a^3 + 6a^2b - 2a^2 - 6ab^2 + 6ab - a - 2b^3 + 2b^2 \geq 0$ through the product of the complex terms mentioned above.

Finally we see that $-2a^3 + 6a^2b - 3a^2 + 6ab^2 - 4ab - 2b^3 + 3b^2 - b = 2b^2(a - b) + 2b^2 + 4ab(b - 1) + b(b - 1) + a^2(6b - 2a - 3)$. Because $a > b \geq 1$ we see that $2b^2(a - b) \geq 0$ $4ab(b - 1) \geq 0$ and $b(b - 1) \geq 0$. To show that $a^2(6b - 2a - 3) \geq 0$ assume that $a = 3b + (n - 2)$ and $n \geq 1$. A contradiction will prove that the term positive. Notice that under this assumption$2(a^2 - b^2) - 4ab + a = 4b^2 + (8(n - 2) + 3)b + (n - 2) + 2(n - 2)^2$. Clearly if $n \geq 2$ this would be positive and this would contradict the assumption that $2(a^2 - b^2) - 4ab + a < 0$. If $n = 1$ then $2(a^2 - b^2) - 4ab + a = 4b^2 - 5b + 1 = (4b - 1)(b - 1) \geq 0$ is another contradiction. We see that $a^2(6b - 2a - 3) \geq 0$. This means that $-2a^3 + 6a^2b - 3a^2 + 6ab^2 - 4ab - 2b^3 + 3b^2 - b \geq 0$ and both the real and imaginary components of the denominator of this unit fraction are positive.

3. REMAINDER ASSOCIATES OF 1

For this class we consider remainders that are conjugates of 1. Two of these cases are trivial and two of these cases are more difficult. We start with the two easy cases.

Remainder -1

We have that $a \geq 1$,$b \geq 0$ and $a > b$.

$$\begin{aligned}
\frac{4}{4(a+bi)-1} &= \frac{1}{a+bi} + \frac{1}{(a+bi)(4(a+bi)-1)} \\
&= \frac{1}{a+bi} + \frac{1}{(a(4a-1)-4b^2) + (4ab + b(4a-1))i} \\
&= \frac{1}{a+bi} + \frac{1}{2(a(4a-1)-4b^2) + 2(4ab + b(4a-1))i} \\
&\qquad + \frac{1}{2(a(4a-1)-b^2) + 2(4ab + b(4a-1))i}.
\end{aligned}$$

Notice that because $a > b$ and both are integers, $a - b \geq 1$. This implies that $8(a^2 - b^2) \geq 8(a + b)$. We see then that $2(a(4a - 1) - 4b^2) = 8(a^2 - b^2) - 2a \geq 8(a + b) - 2a \geq 6a > 0$.

Also notice that $4ab \geq 0$ and $b(4a - 1) \geq 0$. This will imply that $2(4ab + b(4a - 1)) \geq 0$. We have then that $4(a(4a - 1) - b^2)(4ab + b(4a - 1)) \geq 0$.

Remainder 1
We have that $a \geq 0$,$b \geq 0$ and $a > b$.

$$\begin{aligned}\frac{4}{4(a+bi)+1} &= \frac{1}{a+bi} + \frac{1}{(a+bi)(4(a+bi)+1)} \\ &= \frac{1}{a+bi} + \frac{1}{(a(4a+1)-4b^2)+(4ab+b(4a+1))i} \\ &= \frac{1}{a+bi} + \frac{1}{2(a(4a+1)-4b^2)+2(4ab+b(4a+1))i} \\ &\qquad + \frac{1}{2(a(4a+1)-b^2)+2(4ab+b(4a+1))i}.\end{aligned}$$

Notice that because $a > b$ and both are integers, $a - b \geq 1$. This implies that $8(a^2 - b^2) \geq 8(a + b)$. We see then that $2(a(4a + 1) - 4b^2) = 8(a^2 - b^2) + 2a \geq 8(a + b) \geq 0$.

Also notice that $4ab \geq 0$ and $b(4a + 1) \geq 0$. This will imply that $2(4ab + b(4a + 1)) \geq 0$. We have then that $4(a(4a - 1) - b^2)(4ab + b(4a + 1)) \geq 0$.

Remainder $-i$
Recall that it suffices to find a decomposition for $a \geq 0$, $b \geq 0$ and $a > b$. We first assume that $a + b$ is even and $4(a^2 - b^2) - 8ab + (a + b) \leq 0$. We have that

$$\begin{aligned}\frac{4}{4(a+bi)-i} &= \frac{1}{a+bi} + \frac{i}{(a+bi)(4(a+bi)-i)} \\ &= \frac{1}{a+bi} + \frac{i}{(4(a^2-b^2)+b)+(8ab-a)i} \\ &= \frac{1}{a+bi} - \frac{1}{(4(a^2-b^2)+b)+(8ab-a)i} \\ &\qquad + \frac{1+i}{(4(a^2-b^2)+b)+(8ab-a)i}.\end{aligned}$$

Notice that $b \geq 1$ implies that $8ab - a = a(8b-1) \geq 0$. Because $a \geq b$ we see that $4(a^2-b^2)+b \geq 0$ so the second fraction must have both the real and imaginary parts positive.

For the third fraction we multiply the numerator and denominator by $(1-i)$ to find that it is equivalent to

$$\frac{2}{(4(a^2-b^2)+8ab-(a-b))+(-4(a^2-b^2)+8ab-(a+b))i}.$$

Because $a+b$ is even, we have that $a-b$ is even and 2 can be factored from the real and imaginary part of the denominator and canceled to make a unit fraction. Because $4(a^2-b^2)+b \geq 0$ and $8ab-a \geq 0$ implies that their sum $4(a^2-b^2)+8ab-(a-b) \geq 0$. By supposition we have that $-4(a^2-b^2)+8ab-(a+b) \geq 0$. This makes both the real and imaginary part of the third fraction positive.

Next assume that $a+b$ is even and $4(a^2-b^2)-8ab+(a+b) \geq 0$. We have that

$$\begin{aligned}\frac{4}{4(a+bi)-i} &= \frac{1}{a+bi} + \frac{i}{(a+bi)(4(a+bi)-i)} \\ &= \frac{1}{a+bi} + \frac{i}{(4(a^2-b^2)+b)+(8ab-a)i} \\ &= \frac{1}{a+bi} + \frac{1}{(4(a^2-b^2)+b)+(8ab-a)i} \\ &\qquad - \frac{1-i}{(4(a^2-b^2)+b)+(8ab-a)i}.\end{aligned}$$

Notice that $b \geq 1$ implies that $8ab - a = a(8b-1) \geq 0$. Because $a \geq b$ we see that $4(a^2-b^2)+b \geq 0$ so the second fraction must have both the real and imaginary parts positive.

For the third fraction we multiply the numerator and denominator by $(1+i)$ to find that it is equivalent to

$$\frac{2}{(4(a^2-b^2)-8ab+(a+b))+(4(a^2-b^2)+8ab-(a-b))i}.$$

Because $a+b$ is even, we have that $a-b$ is even and 2 can be factored from the real and imaginary part of the denominator and canceled to make a unit fraction. Because $4(a^2-b^2)+b \geq 0$ and $8ab-a \geq 0$ implies that their sum $4(a^2-b^2)+8ab-(a-b) \geq 0$. By supposition we have that $4(a^2-b^2)-8ab+(a+b) \geq 0$. This makes both the real and imaginary part of the third fraction positive.

NEEDS MORE WORK

Assume that $a+b$ is odd, $2 \equiv a \mod 3$ and $1 \equiv b \mod 3$. It particular suppose that there exist nonnegative integers k and ℓ so that $a = 3k+2$ and $b = 3\ell+1$. We can see that because $a+b$ is odd, then $k+\ell$ is even. Furthermore, because $a \geq b$ we can see that $k \geq \ell$. We see that

$$\begin{aligned}\frac{4}{4(a+bi)-i} &= \frac{4}{(12k+8)+(12\ell+3)i} \\ &= \frac{1}{(4k+3)+(4\ell+1)i} \\ &\quad + \frac{(4k+4)+(4\ell+1)i}{((12k+8)+(12\ell+3)i)((4k+3)+(4\ell+1)i)} \\ &= \frac{1}{(4k+3)+(4\ell+1)i} + \frac{1}{(12k+8)+(12\ell+3)i} \\ &\quad + \frac{1}{((12k+8)+(12\ell+3)i)((4k+3)+(4\ell+1)i)}\end{aligned}$$

We see that the last fraction becomes

$$\frac{1}{(48(k^2-\ell^2)+4(17k-6\ell)+21)+(96k\ell+24k+68\ell+17)i}$$

Because $k \geq \ell \geq 0$ we see that both the real and imaginary parts of the third fraction are positive.

Next assume that $a+b$ is odd, $0 \equiv a \mod 3$ and $0 \equiv b \mod 3$. In particular suppose that there exist nonnegative integers k and ℓ so that $a = 3k$ and $b = 3\ell$. We can see that because $a+b$ is odd, then $k+\ell$ is odd. Furthermore, because $a \geq b$ we can see that $k \geq \ell$. We see that

$$\frac{4}{4(a+bi)-i} = \frac{4}{12k+(12\ell-1)i}.$$

Remainder i

Recall that it suffices to find a decomposition for $a \geq 0$, $b \geq 0$ and $a > b$. We first assume that $a+b$ is even and $4(a^2-b^2)-8ab-(a+b) \leq 0$. We have that

$$\begin{aligned}\frac{4}{4(a+bi)+i} &= \frac{1}{a+bi} - \frac{i}{(a+bi)(4(a+bi)+i)} \\ &= \frac{1}{a+bi} - \frac{i}{(4(a^2-b^2)-b)+(8ab+a)i} \\ &= \frac{1}{a+bi} + \frac{1}{(4(a^2-b^2)-b)+(8ab+a)i} \\ &\qquad - \frac{1+i}{(4(a^2-b^2)-b)+(8ab+a)i}.\end{aligned}$$

Notice that $a > b$ implies that $4(a^2-b^2)-b = 4a(a-b)+b(4(a-b)-1) \geq 0$. Clearly $8ab+a \geq 0$ so the second fraction must have both the real and imaginary parts positive.

For the third fraction we multiply the numerator and denominator by $(1-i)$ to find that it is equivalent to

$$\frac{2}{(4(a^2-b^2)+8ab+(a-b))+(-4(a^2-b^2)+8ab+(a+b))i}.$$

Because $a+b$ is even, we have that $a-b$ is even and 2 can be factored from the real and imaginary part of the denominator and canceled to make a unit fraction. Because $4(a^2-b^2)-b \geq 0$ and $8ab+a \geq 0$ implies that their sum $4(a^2-b^2)+8ab+(a-b) \geq 0$. By supposition we have that $-4(a^2-b^2)+8ab+(a+b) \geq 0$. This makes both the real and imaginary part of the third fraction positive.

Next assume that $a+b$ is even and $4(a^2-b^2)-8ab-(a+b) \geq 0$. We have that

$$\begin{aligned}\frac{4}{4(a+bi)+i} &= \frac{1}{a+bi} - \frac{i}{(a+bi)(4(a+bi)+i)} \\ &= \frac{1}{a+bi} - \frac{i}{(4(a^2-b^2)-b)+(8ab+a)i} \\ &= \frac{1}{a+bi} - \frac{1}{(4(a^2-b^2)-b)+(8ab+a)i} \\ &\qquad + \frac{1-i}{(4(a^2-b^2)-b)+(8ab+a)i}.\end{aligned}$$

Notice that $a > b$ implies that $4(a^2-b^2)-b = 4a(a-b)+b(4(a-b)-1) \geq 0$. Clearly $8ab+a \geq 0$ so the second fraction must have both the real and imaginary parts positive.

For the third fraction we multiply the numerator and denominator by $(1+i)$ to find that it is equivalent to

$$\frac{2}{(4(a^2-b^2)-8ab-(a+b))+(4(a^2-b^2)+8ab+(a-b))i}.$$

Because $a+b$ is even, we have that $a-b$ is even and 2 can be factored from the real and imaginary part of the denominator and canceled to make a unit fraction. Because $4(a^2-b^2)-b \geq 0$ and $8ab+a \geq 0$ implies that their sum $4(a^2-b^2)+8ab+(a-b) \geq 0$. By supposition we have that $4(a^2-b^2)-8ab-(a+b) \geq 0$. This makes both the real and imaginary part of the third fraction positive.

NEEDS MORE WORK
Next assume that $b=0$ and $a \geq 1$. We have that

$$\frac{4}{4a+i} = \frac{1}{a} - \frac{i}{4a^2+ai}$$

$$= \frac{1}{a} + \frac{1}{4a^2 i} - \left(\frac{1}{4a^2 i} + \frac{i}{4a^2 + ai} \right)$$
$$= \frac{1}{a} + \frac{1}{4a^2 i} - \frac{1}{16a^3 + 4a^2 i}$$

Next assume that $a = n \cdot b$ where $n \geq 1$ and $b \geq 1$.

$$\frac{4}{4nb + (4b+1)i} = \frac{1}{b(n+i)} - \frac{i}{b((4b(n^2-1)-1) + (8nb+n)i)}$$

4. REMAINDER ASSOCIATES OF $1 + 2I$

Remainder $1 + 2i$

Remainder $1 - 2i$

Remainder $2 + i$
Assume that $a + b$ is even and $4(a^2 - b^2) - 8ab + a - 3b \leq 0$. We see that

$$\begin{aligned}
\frac{4}{4(a+bi) + (2+i)} &= \frac{1}{a+bi} - \frac{2+i}{(a+bi)(4(a+bi)+(2+i))} \\
&= \frac{1}{a+bi} - \frac{2+i}{(4(a^2-b^2)+2a-b) + (8ab+a+2b)i} \\
&= \frac{1}{a+bi} - \frac{1}{(4(a^2-b^2)+2a-b) + (8ab+a+2b)i} \\
&\quad - \frac{1+i}{(4(a^2-b^2)+2a-b) + (8ab+a+2b)i}.
\end{aligned}$$

We see that

$$\frac{2}{(4(a^2-b^2)+8ab+3a+b) + (-4(a^2-b^2)+8ab-a+3b)i}.$$

Assume that $a + b$ is even. We see that

$$\begin{aligned}
\frac{4}{4((a+1)+bi) - (2-i)} &= \frac{1}{(a+1)+bi} + \frac{2-i}{((a+1)+bi)(4((a+1)+bi)-(2-i))} \\
&= \frac{1}{(a+1)+bi} + \frac{2-i}{(4(a^2-b^2)+6a-b+2) + (8ab+a+6b+1)i} \\
&= \frac{1}{(a+1)+bi} + \frac{1}{(4(a^2-b^2)+6a-b+2) + (8ab+a+6b+1)i} \\
&\quad + \frac{1-i}{(4(a^2-b^2)+6a-b+2) + (8ab+a+6b+1)i}.
\end{aligned}$$

We see that

$$\frac{2}{(4(a^2-b^2)-8ab+5a-7b+1)+(4(a^2-b^2)+8ab+7a+5b+3)i}$$

Remainder $2-i$

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