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AUTOMORPHISMS OF NORMAL TRANSFORMATION SEMIGROUPS

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1. Introduction and preliminaries

Let X be an infinite set, $\mathscr{G}_X$ be the group of all bijections of X and S be a semigroup of total transformations of X with the composition of transformations f and g in S defined by the formula

$$fg(x)=f(g(x)), \quad \text{where} \quad x\in X.$$

We say that S is a $\mathscr{G}_X$-*normal* semigroup if

$$hSh^{-1}=S, \quad \text{for all} \quad h\in\mathscr{G}_X.$$

The full transformation semigroup T_X, the semigroups of all 1–1 and all onto transformations and the group $\mathscr{G}_X$ itself, are examples of $\mathscr{G}_X$-normal semigroups.

If S is a $\mathscr{G}_X$-normal semigroup, then for each $h\in\mathscr{G}_X$, the map ϕ of S given by

$$\phi(f)=hfh^{-1} \ (f\in S)$$

is an automorphism of S, specifically an *inner automorphism* of S. Our purpose is to prove the following:

Theorem 1.1. *Every automorphism of a $\mathscr{G}_X$-normal semigroup is inner.*

The subject of this paper was suggested to the author by G. R. Wood.

The question of whether inner automorphisms exhaust all automorphisms of a $\mathscr{G}_X$-normal semigroup has attracted the attention of a number of authors. In 1937 Schreier [**10**] was the first to give a positive answer for T_X. Then Malcev [**6**] extended this result to every ideal of T_X. Next Sullivan [**12**] generalized this work and confirmed that if a semigroup contains all constant transformations (in particular if a $\mathscr{G}_X$-normal semigroup contains a constant transformation) then it possesses only inner automorphisms, while Fitzpatrick and Symons [**3**] showed this for a semigroup containing $\mathscr{G}_X$. Schein [**8,9**] discovered that a $\mathscr{G}_X$-normal semigroup of 1–1 transformations has only inner automorphisms (see [**4**] for the special case of Baer–Levi semigroups).

Our result subsumes all previously stated results for $\mathscr{G}_X$-normal semigroups and describes completely all automorphisms of every $\mathscr{G}_X$-normal transformation semigroup.

In this paper we use a technique which differs from those used by Sullivan [**12**] and Schein [**8, 9**]. The essence is the production of certain maximal right (Section 2) and left (Section 3) ideals. We note a remarkable duality between properties of these right and left ideals.

For the purpose of our proof we partition all $\mathscr{G}_X$-normal semigroups into three types:

1. Semigroups containing a constant map; and constant-free semigroups into:
2. Semigroups of 1–1 transformations; and
3. Constant-free semigroups containing a transformation which is not 1–1.

All automorphisms of semigroups of the first type are inner [**12**, Theorem 1], so we can restrict our attention to constant-free semigroups.

We begin with some general notes on $\mathscr{G}_X$-normal semigroups.

For a function $f: X \to X$ we denote the *range* of f by $R(f)$ $(=f(X))$ and the *partition* of f by $\pi(f)$ $(=\{f^{-1}(x): x \in R(f)\})$.

If S is an arbitrary semigroup of transformations, let

$$R(S)=\{R(f): f \in S\} \quad \text{and} \quad \pi(S)=\{\pi(f): f \in S\}.$$

We say that $R(S)$ ($\pi(S)$) is *normal* if for each $h \in \mathscr{G}_X$

$$h(R(S))=R(S) \quad (h(\pi(S))=\pi(S)),$$

(by $h(R(S))$ we mean $\{h(A): A \in R(S)\}$ and by $h(\pi(S))$ we mean $\{h(\mathscr{A}): \mathscr{A} \in \pi(S)\}$, where $h(\mathscr{A})=\{h(A): A \in \mathscr{A}\}$).

Lemma 1.2. *If S is a $\mathscr{G}_X$-normal semigroup, then $R(S)$ and $\pi(S)$ are normal.*

The proof is straightforward. □

We say that a semigroup S is *trivial* if $S=\{\Delta_X\}$, where Δ_X is the identity transformation of X. In what follows S is non-trivial.

Result 1.3. *Every $\mathscr{G}_X$-normal semigroup S is transitive.*

Proof. Take arbitrary x, y in X. We construct f in S such that $f(x)=y$.

Firstly let x and y be distinct and suppose there exists a $g \in S$ with $g(x)=z \neq x$. If $z=y$ we let $f=g$, otherwise $(y,z)g(y,z)$ is the required f ((y,z) denotes the transposition interchanging y and z). To construct g, observe that since S is non-trivial there exists a $q \in S$ together with distinct u and v in X such that $q(u)=v$. If $u=x$ we let $g=q$, otherwise $g=(u,x)q(u,x)$.

Now suppose $y=x$, choose any p in S and let $p(x)=w$. If $w=x$ we let $f=p$. Otherwise choose $t \in S$ with $t(w)=x$ (using the first part of the proof), then $f=tp$ takes x to x as required. □

Remark 1.4. We exclude from our consideration $\mathscr{G}_X$-normal subsemigroups of $\mathscr{G}_X$, since they are all subgroups of $\mathscr{G}_X$, and hence have only inner automorphisms [**11**].

2. $\mathscr{G}_X$-normal semigroups of 1–1 transformations

In this section S denotes a $\mathscr{G}_X$-normal semigroup of 1–1 transformations.

Definition 2.1. Let $x \in X$ and

$$\mathscr{R}_x = \{r \in S : x \in X \backslash R(r)\}.$$

Then $\mathscr{R}_x$ is a right ideal of S, which we call a *point right ideal.* □

We will use the following observation based on the normality of $R(S)$ (Lemma 1.2) and the fact that S is not a subsemigroup of $\mathscr{G}_X$, that is $R(S)$ contains proper subsets of X.

Remark 2.2. Given $x, y \in X$ with $x \neq y$ there exists an A in $R(S)$ with $x \in X \backslash A$ and $y \in A$. □

Lemma 2.3. *Given $x, y \in X$ the following three statements are equivalent:*

(i) $\mathscr{R}_x \subseteq \mathscr{R}_y$;
(ii) $x = y$;
(iii) $\mathscr{R}_x = \mathscr{R}_y$.

Proof. Implications (ii)⇒(iii) and (iii)⇒(i) are trivial. We show (i)⇒(ii). Suppose $x \neq y$ and choose an $A \in R(S)$ with $x \in X \backslash A$, $y \in A$ (Remark 2.2). If $f \in S$ with $R(f) = A$, then $f \in \mathscr{R}_x \backslash \mathscr{R}_y$, so $\mathscr{R}_x \nsubseteq \mathscr{R}_y$, proving (i)⇒(ii). □

Define a map $\theta : X \to \{\mathscr{R}_x : x \in X\}$ via $\theta(x) = \mathscr{R}_x$, each $x \in X$.

Lemma 2.4. *θ is a bijection.*

Proof. Clearly θ is onto and Lemma 2.3 ensures θ is 1–1. □

Definition 2.5. Given distinct $f_1, f_2 \in S$ let

$$\mathscr{R}_{f_1, f_2} = \{r \in S : f_1 r = f_2 r\}.$$

Then $\mathscr{R}_{f_1, f_2}$ is a right ideal of S (possibly empty), which we call a *function right ideal.* □

We will show (Result 2.8) that there always exist distinct f_1, f_2 in S such that $\mathscr{R}_{f_1, f_2}$ is non-empty. However $\mathscr{R}_{f_1, f_2}$ may be empty. Observe that given f_1 and f_2,

$$r \in \mathscr{R}_{f_1, f_2} \quad \text{iff} \quad R(r) \subseteq \{x \in X : f_1(x) = f_2(x)\}.$$

Hence if we choose f_1 and f_2 which are never equal, then $\mathscr{R}_{f_1, f_2} = \Phi$.

Let S, for example, be the Baer–Levi semigroup of type $(|X|, |X|)$ [2], that is the semigroup of all 1–1 transformations f such that $|R(f)| = |X \backslash R(f)| = |X|$. Note that S is

$\mathscr{G}_X$-normal and choose $f_1 \in S$, then $X \backslash R(f_1) \in R(S)$ (Lemma 1.2). If $f_2 \in S$ with $R(f_2) = X \backslash R(f_1)$, then $\mathscr{R}_{f_1, f_2} = \Phi$.

The following notation applies to an arbitrary $\mathscr{G}_X$-normal semigroup S.

Notation 2.6. Let f_1, f_2 be distinct transformations in S. Then

$$\mathscr{D}_{f_1, f_2} = \{x \in X : f_1(x) \neq f_2(x)\}$$

and

$$D_{f_1, f_2} = \{\{f_1(x), f_2(x)\} : x \in \mathscr{D}_{f_1, f_2}\}.$$ □

Returning to semigroups of 1–1 transformations, we now derive relationships between point right ideals and function right ideals.

Result 2.7. *Let $f_1, f_2 \in S$ with $\mathscr{R}_{f_1, f_2} \neq \Phi$. Then*

$$\mathscr{R}_{f_1, f_2} = \bigcap_{x \in \mathscr{D}_{f_1, f_2}} \mathscr{R}_x.$$

Proof. Let $r \in \mathscr{R}_{f_1, f_2}$, that is $f_1 r = f_2 r$. If $x \in \mathscr{D}_{f_1, f_2}$, or $f_1(x) \neq f_2(x)$, then $x \in X \backslash R(r)$, so $r \in \mathscr{R}_x$, and since this is true for each $x \in \mathscr{D}_{f_1, f_2}$ we conclude

$$r \in \bigcap_{x \in \mathscr{D}_{f_1, f_2}} \mathscr{R}_x$$

or

$$\mathscr{R}_{f_1, f_2} \subseteq \bigcap_{x \in \mathscr{D}_{f_1, f_2}} \mathscr{R}_x.$$

Conversely, if

$$r \in \bigcap_{x \in \mathscr{D}_{f_1, f_2}} \mathscr{R}_x,$$

then for each y in $R(r)$ we have $y \in X \backslash \mathscr{D}_{f_1, f_2}$, or $f_1(y) = f_2(y)$ and hence $f_1 r = f_2 r$, that is $r \in \mathscr{R}_{f_1, f_2}$, so

$$\bigcap_{x \in \mathscr{D}_{f_1, f_2}} \mathscr{R}_x \subseteq \mathscr{R}_{f_1, f_2},$$

which proves the desired equality. □

Result 2.8. *Given $x \in X$ there exist $f_1, f_2 \in S$ such that $\mathscr{R}_x = \mathscr{R}_{f_1, f_2}$.*

Proof. On account of Result 2.7 it is sufficient to construct f_1, f_2 such that $\mathscr{D}_{f_1, f_2} = \{x\}$.

Observe that there exists an f in S with

$$|X \backslash R(f)| \geqq 2.$$

(For an arbitrary f in $S\backslash\mathcal{G}_X$

$$|X\backslash R(f^2)|=|X\backslash R(f)|+|X\backslash R(f)|$$

and we replace f with f^2).

Using the normality of $R(S)$ (Lemma 1.2) choose an f in S with

$$x\in X\backslash R(f) \quad \text{and} \quad |X\backslash R(f)|\geqq 2.$$

Let $f(x)=y$ and $z\in X\backslash R(f)$, $z\neq x$. If

$$g=(x,z)f(x,z)$$

then $g(z)=y$ and $z\in X\backslash R(g)$. We let

$$h=(y,z),\ f_1=gf \text{ and } \quad f_2=hgh^{-1}f.$$

Then for each $u\neq x$:

$$\begin{aligned} f_1(u)=gf(u)=gh^{-1}f(u), &\quad \text{since} \quad f(u)\neq y \quad \text{for} \quad u\neq x \\ &\quad \text{and} \quad z\notin R(f); \\ =hgh^{-1}f(u), &\quad \text{since} \quad gh^{-1}f(u)\neq y \\ &\quad \text{for} \quad f(u)\neq y \\ &\quad \text{and} \quad z\notin R(g); \\ =f_2(u). & \end{aligned}$$

However

$$f_1(x)=gf(x)=g(y)$$

while

$$f_2(x)=hgh^{-1}f(x)=hgh^{-1}(y)=hg(z)=h(y)=z\neq g(y),$$

since $z\in X\backslash R(g)$. Hence $f_1(x)\neq f_2(x)$ and $\mathcal{D}_{f_1,f_2}=\{x\}$. □

Result 2.9. *Given f_1 and f_2 in S, $\mathcal{R}_{f_1,f_2}$ is a maximal function right ideal if and only if $|\mathcal{D}_{f_1,f_2}|=1$.*

Proof. Suppose $\mathcal{R}_{f_1,f_2}$ is a maximal function right ideal, while $x,y\in\mathcal{D}_{f_1,f_2}$, $x\neq y$.

Then

$$\begin{aligned}\mathscr{R}_{f_1,f_2} &= \bigcap_{z\in\mathscr{D}_{f_1,f_2}} \mathscr{R}_z \qquad \text{(Result 2.7)}\\ &\subseteqq \mathscr{R}_x \cap \mathscr{R}_y\\ &\subsetneqq \mathscr{R}_x \qquad \text{(Lemma 2.3).}\end{aligned}$$

It follows from Result 2.8 that there exist g_1 and g_2 with

$$\mathscr{R}_{g_1,g_2} = \mathscr{R}_x,$$

and so

$$\mathscr{R}_{f_1,f_2} \subsetneqq \mathscr{R}_x = \mathscr{R}_{g_1,g_2},$$

a contradiction to the maximality of $\mathscr{R}_{f_1,f_2}$. Hence $|\mathscr{D}_{f_1,f_2}| = 1$.

For the converse, suppose $\mathscr{D}_{f_1,f_2} = \{x\}$, some $x \in X$, while there exist $g_1, g_2 \in S$ such that

$$\mathscr{R}_{g_1,g_2} \supseteq \mathscr{R}_{f_1,f_2}.$$

Since

$$\mathscr{R}_{g_1,g_2} = \bigcap_{y\in\mathscr{D}_{g_1,g_2}} \mathscr{R}_y \qquad \text{(Result 2.7)}$$

we have

$$\bigcap_{y\in\mathscr{D}_{g_1,g_2}} \mathscr{R}_y = \mathscr{R}_{g_1,g_2} \supseteq \mathscr{R}_{f_1,f_2} = \mathscr{R}_x \qquad \text{(Result 2.7 again),}$$

and so Lemma 2.3 ensures $\mathscr{D}_{g_1,g_2} = \{x\}$, that is

$$\mathscr{R}_{g_1,g_2} = \mathscr{R}_x = \mathscr{R}_{f_1,f_2}. \qquad \square$$

Corollary 2.10. *Given f_1 and f_2 in S, $\mathscr{R}_{f_1,f_2}$ is a maximal function right ideal if and only if $\mathscr{R}_{f_1,f_2} = \mathscr{R}_x$, some $x \in X$.*

Proof. Follows from Results 2.7 and 2.9. $\square$

We show now that each automorphism ϕ of S permutes point right ideals.

Result 2.11. *Given $x \in X$,*

$$\phi(\mathscr{R}_x) = \mathscr{R}_y,$$

for some $y \in X$.

Proof. Choose f_1 and f_2 in S such that $\mathscr{R}_{f_1,f_2}=\mathscr{R}_x$ (Result 2.8), then

$$\begin{aligned}\phi(\mathscr{R}_x)=\phi(\mathscr{R}_{f_1,f_2})&=\phi(\{r: f_1r=f_2r\})\\&=\{\phi(r): \phi(f_1r)=\phi(f_2r)\}\\&=\{\phi(r): \phi(f_1)\phi(r)=\phi(f_2)\phi(r)\}\\&=\{r': \phi(f_1)r'=\phi(f_2)r'\}\\&=\mathscr{R}_{\phi(f_1),\phi(f_2)}.\end{aligned}$$

Now Corollary 2.10 ensures $\mathscr{R}_{f_1,f_2}$ is a maximal function right ideal, hence $\mathscr{R}_{\phi(f_1),\phi(f_2)}(=\phi(\mathscr{R}_{f_1,f_2}))$ is a maximal function right ideal, so there exists $y\in X$ such that

$$\mathscr{R}_{\phi(f_1),\phi(f_2)}=\mathscr{R}_y \qquad \text{(Corollary 2.10)}$$

and thus

$$\phi(\mathscr{R}_x)=\mathscr{R}_{\phi(f_1),\phi(f_2)}=\mathscr{R}_y. \qquad \square$$

Define a map

$$\eta: \{\mathscr{R}_x: x\in X\}\to\{\mathscr{R}_x: x\in X\}$$

via $\eta(\mathscr{R}_x)=\phi(\mathscr{R}_x)$, each $\mathscr{R}_x\subseteq S$.

Lemma 2.12. *η is a bijection.*

Proof. That η is a mapping is the content of Result 2.11. Similarly by considering the automorphism ϕ^{-1} we define a map

$$\zeta: \{\mathscr{R}_x: x\in X\}\to\{\mathscr{R}_x: x\in X\}$$

via $\zeta(\mathscr{R}_x)=\phi^{-1}(\mathscr{R}_x)$, each $\mathscr{R}_x\subseteq S$.
Certainly, ζ is the inverse of η and so η is a bijection. $\square$

We now define a map

$$h: X\to X \quad \text{via} \quad h(x)=y, \quad \text{where} \quad \eta(\mathscr{R}_x)=\mathscr{R}_y, \quad \text{each} \quad x\in X.$$

It is clear, that

$$h=\theta^{-1}\eta\theta,$$

and so Lemmas 2.4 and 2.12 ensure h is a bijection of X. We call h the *bijection associated with ϕ*.

Lemma 2.13. *Given $f \in S$,*

$$R(\phi(f)) = h(R(f)).$$

Proof. Observe that to show $R(\phi(f)) = h(R(f))$ it is sufficient to show that

$$X \backslash R(\phi(f)) = h(X \backslash R(f)),$$

because for the bijection h, $h(X \backslash R(f)) = X \backslash h(R(f))$.

Now if $x \in X \backslash R(f)$, that is $f \in \mathscr{R}_x$, then $\phi(f) \in \eta(\mathscr{R}_x) = \mathscr{R}_{h(x)}$, so $h(x) \in X \backslash R(\phi(f))$, or

$$h(X \backslash R(f)) \subseteq X \backslash R(\phi(f)).$$

To show the reverse inclusion is true, observe that $h^{-1} = \theta^{-1}\eta^{-1}\theta$ is the bijection associated with ϕ^{-1} and so the first part of the proof implies that given $g \in S$,

$$h^{-1}(X \backslash R(g)) \subseteq X \backslash R(\phi^{-1}(g)).$$

In particular taking $g = \phi(f)$ we have $h^{-1}(X \backslash R(\phi(f))) \subseteq X \backslash R(\phi^{-1}(\phi(f)))$, or

$$h(X \backslash R(f)) \supseteq X \backslash R(\phi(f)),$$

and the equality follows. □

We complete our study of automorphisms of $\mathscr{G}_X$-normal semigroups of 1–1 transformations, that is, semigroups of Type 2, by presenting the following result.

Result 2.14. *Let S be a $\mathscr{G}_X$-normal semigroup of 1–1 transformations ($S \nsubseteq \mathscr{G}_X$). Then each automorphism ϕ of S is inner, that is, for some $h \in \mathscr{G}_X$*

$$\phi(f) = hfh^{-1}, \quad \textit{for each} \quad f \in S.$$

Proof. Consider the bijection h associated with ϕ as defined prior to Lemma 2.13. Take an arbitrary $f \in S$, $x \in X$ and let $f(x) = y$. Choose A in $R(S)$ with $A \neq X$ and $x \in A$. Let $z \in X \backslash A$ and $B = (A \backslash \{x\}) \cup \{z\} \in R(S)$ (Lemma 1.2). Choose p and q in S such that $R(p) = A$ and $R(q) = B$.

Now $R(p) \backslash R(q) = A \backslash B = \{x\}$, thus $R(fp) \backslash R(fq) = \{f(x)\} = \{y\}$. Using Lemma 2.13 we have:

$$R(\phi(p)) \backslash R(\phi(q)) = \{h(x)\}$$

and

$$R(\phi(fp)) \backslash R(\phi(fq)) = \{h(y)\}.$$

However

$$\begin{aligned} R(\phi(fp)) \backslash R(\phi(fq)) &= R(\phi(f)\phi(p)) \backslash R(\phi(f)\phi(q)) \\ &= \{\phi(f)h(x)\}, \end{aligned}$$

so

$$\phi(f)h(x)=h(y)=hf(x), \quad \text{that is}$$

$$\phi(f)=hfh^{-1}. \qquad \square$$

Remark 2.15. The fact that every $\mathcal{G}_X$-normal semigroup of 1–1 transformations possesses only inner automorphisms was first established by B. M. Schein [**8, 9**]. We understand that his proof, based on the study of ordered sets of ranges, is quite different from ours. □

3. $\mathcal{G}_X$-normal constant-free semigroups containing a transformation which is not 1–1

Let S be a $\mathcal{G}_X$-normal constant-free semigroup containing a transformation which is not 1–1. We prove that all automorphisms of S are inner. We start by showing that $R(S)$ contains only sets of cardinality $|X|$.

Lemma 3.1. *If S is a $\mathcal{G}_X$-normal constant-free semigroup, then $|R(f)|=|X|$, each $f\in S$.*

Proof. Suppose there is an f in S with $|R(f)|=\alpha<|X|$, that is $|\pi(f)|=|R(f)|=\alpha$. We show that there exists an $A\in\pi(f)$ with $|A|\geqq\alpha$. The result is clear when α is finite. Hence assume α is infinite and denote by α^+ the cardinal successor of α. Then either $\alpha^+=|X|$ (and so $|X|$ is regular [7, 21.14]) or there exists $\beta<|X|$, $\beta=\alpha^+$ (and so β is regular [7, 21.14]). The assumption that each $A\in\pi(f)$ has a cardinality less than α implies that $|\cup\pi(f)|<|X|$ or $|\cup\pi(f)|<\beta<|X|$ respectively [7, 21.18], a contradiction. Hence we can choose an $A\in\pi(f)$ with $|A|\geqq\alpha$ and a $B\in R(S)$ with $B\subseteq A$ and $|B|=\alpha$ (Lemma 1.2) together with a $g\in S$ such that $R(g)=B$. Then $|R(fg)|=1$, so that fg is a constant map in S, a contradiction which proves $|R(f)|=|X|$. □

Let $\mathcal{P}_2$ be the set of all doubletons in X.

Definition 3.2. Given $A\in\mathcal{P}_2$, $A=\{a_1,a_2\}$, let

$$\mathcal{L}_A=\{l\in S: l(a_1)=l(a_2)\}.$$

Then $\mathcal{L}_A$ is a left ideal of S which we call a *point left ideal.* □

Lemma 3.3. *For each $A\in\mathcal{P}_2$, $\mathcal{L}_A\neq\Phi$.*

Proof. Choose a map f in S which is not 1–1, say $f(x)=f(y)$ for distinct $x,y\in X$. If $h\in\mathcal{G}_X$ is such that $\{h(x),h(y)\}=A$ then $hfh^{-1}\in\mathcal{L}_A$. □

Lemma 3.4. *Given $A, B \in \mathscr{P}_2$, the following three statements are equivalent:*

(i) $\mathscr{L}_A \subseteq \mathscr{L}_B$;

(ii) $A = B$;

(iii) $\mathscr{L}_A = \mathscr{L}_B$.

Proof. Implications (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(i) are trivial. We show (i)$\Rightarrow$(ii).

Let $B = \{b_1, b_2\}$ and suppose $A \neq B$, say $b_1 \in B \backslash A$. Choose an $l \in \mathscr{L}_A$ (Lemma 3.3) and let $x \in R(l) \backslash l(A \cup B)$ (note: $|X| = |R(l)| > |l(A \cup B)|$, Lemma 3.1). If $y \in X$ is such that $l(y) = x$, let $h = (b_1, y)$ and $f = hlh^{-1}$. We show $f \in \mathscr{L}_A \backslash \mathscr{L}_B$. That $f \in \mathscr{L}_A$ follows from the fact that h moves only points b_1 and y, which are not in A. To show that $f \notin \mathscr{L}_B$, observe that $f(b_1) = hlh^{-1}(b_1) = hl(y) = h(x)$, while $f(b_2) = hlh^{-1}(b_2) = hl(b_2)$, because $b_2 \neq y$ (else $x = l(y) = l(b_2) = l(B)$, contrary to the choice of x). Hence $f(b_2) \neq h(x)$, because $l(b_2) = l(B) \neq x$. Thus $f(b_1) \neq f(b_2)$ and $f \in \mathscr{L}_B$.

Define a map $\delta: \mathscr{P}_2 \rightarrow \{\mathscr{L}_A : A \in \mathscr{P}_2\}$ via $\delta(A) = \mathscr{L}_A$, each $A \in \mathscr{P}_2$.

Lemma 3.5. *δ is a bijection.*

Proof. Clearly δ is onto and Lemma 3.4 ensures δ is 1–1. □

Definition 3.6. Given distinct f_1, $f_2 \in S$ let

$$\mathscr{L}_{f_1, f_2} = \{l \in S : lf_1 = lf_2\}.$$

Then $\mathscr{L}_{f_1, f_2}$ is a left ideal of S (possibly empty, see Example 3.7 below), which we call a *function left ideal.* □

We will show (Result 3.10) that for each $\mathscr{G}_X$-normal constant-free semigroup S containing a transformation which is not 1–1 there exist $f_1, f_2 \in S$ with $\mathscr{L}_{f_1, f_2} \neq \Phi$. In general, the question of whether f_1, $f_2 \in S$ generate a non-empty $\mathscr{L}_{f_1, f_2}$ is the question of whether the equation $lf_1 = lf_2$ has a solution l in S. The example below illustrates that $\mathscr{L}_{f_1, f_2}$ may be empty.

Example 3.7. Let S be the dual Baer-Levi semigroup of the type $(|X|, |X|)$ **[1]**, that is the semigroup of all onto mappings f such that $|f^{-1}(x)| = |X|$ for each $x \in X$. Certainly S is $\mathscr{G}_X$-normal. Assume $X = \mathbb{N}$, so that $|X| = \aleph_0$. Fix an arbitrary $f_1 \in S$ and let

$$\mathscr{A} = \pi(f_1) = \{A_1, A_2, A_3, \ldots\}.$$

Partition each $A_i \in \mathscr{A}$ such that $A_i = A_i' \dot{\cup} A_i''$, $|A_i'| = A_i''| = \aleph_0$. Let $\mathscr{B}$ be the partition of X

given by

$$\mathscr{B}=\{A'_1, A''_1 \dot{\cup} A'_2, A''_2 \dot{\cup} A'_3, \dots\}.$$

Since $\mathscr{B}$ is a partition of X into $\aleph_0$ sets, each of cardinality $\aleph_0$, $\mathscr{B}\in\pi(S)$, and so there exists $f_2\in S$ with $\pi(f_2)=\mathscr{B}$. Suppose $l\in\mathscr{L}_{f_1,f_2}$ that is $lf_1=lf_2$ and let $l_1(A_1)=x$. Then because of the choice of $\mathscr{B}$ we have the following chain of equalities:

$$x=lf_1(A_1)=lf_1(A''_1)=lf_2(A''_1)=lf_2(A'_2)=lf_1(A'_2)=lf_1(A_2)=\dots$$

thus

$$x=lf_1(A_1)=lf_1(A_2)=\dots,$$

that is $R(lf_1)=\{x\}$ and lf_1 is a constant in S, contradicting the construction of S, so that $\mathscr{L}_{f_1,f_2}=\Phi$. □

Recall that $\mathscr{D}_{f_1,f_2}$ and D_{f_1,f_2} (Notation 2.6) were defined for an arbitrary $\mathscr{G}_X$-normal semigroup S ($f_1, f_2\in S$). The following remark is an immediate consequence of the definition of D_{f_1,f_2}.

Remark 3.8. Let $f_1, f_2\in S$, then $D_{f_1,f_2}\subseteq\mathscr{P}_2$. □

We proceed with two results deriving relationships between point left ideals and function left ideals.

Result 3.9. *Let f_1 and f_2 be distinct elements of S, and $\mathscr{L}_{f_1,f_2}\neq\Phi$. Then*

$$\mathscr{L}_{f_1,f_2}=\bigcap_{A\in D_{f_1,f_2}}\mathscr{L}_A.$$

Proof. Let $l\in\mathscr{L}_{f_1,f_2}$, that is, $lf_1=lf_2$ and so for each $x\in\mathscr{D}_{f_1,f_2}$ we have $lf_1(x)=lf_2(x)$ (recall $f_1(x)\neq f_2(x)$) so $l\in\mathscr{L}_{\{f_1(x),f_2(x)\}}$ and since this is true for each $x\in\mathscr{D}_{f_1,f_2}$ we conclude

$$l\in\bigcap_{x\in\mathscr{D}_{f_1,f_2}}\mathscr{L}_{\{f_1(x),f_2(x)\}}=\bigcap_{A\in D_{f_1,f_2}}\mathscr{L}_A,\quad\text{or}\quad\mathscr{L}_{f_1,f_2}\subseteq\bigcap_{A\in D_{f_1,f_2}}\mathscr{L}_A.$$

Conversely, let $l\in\bigcap_{A\in D_{f_1,f_2}}\mathscr{L}_A$, then for each $x\in\mathscr{D}_{f_1,f_2}$, $lf_1(x)=lf_2(x)$. Now for each $y\notin\mathscr{D}_{f_1,f_2}$ we have $f_1(y)=f_2(y)$, so we deduce $lf_1=lf_2$. That is,

$$l\in\mathscr{L}_{f_1,f_2}\quad\text{and}\quad\bigcap_{A\in D_{f_1,f_2}}\mathscr{L}_A\subseteq\mathscr{L}_{f_1,f_2},$$

which proves the desired equality. □

Result 3.10 *Given an $A \in \mathscr{P}_2$, there exist f_1 and f_2 is S such that*

$$\mathscr{L}_A = \mathscr{L}_{f_1, f_2}.$$

Proof. On account of Result 3.9 it is sufficient to construct f_1 and f_2 such that

$$D_{f_1, f_2} = \{A\}.$$

Choose an f in $\mathscr{L}_A$ (Lemma 3.3) and let $f(A) = z$. Let $A = \{a_1, a_2\}$. Since S is transitive (Result 1.3) there exists g in S such that $g(z) = a_1$. Let $h = (a_1, a_2)$ and

$$f_1 = gf; \quad f_2 = hf_1h^{-1}.$$

Since h moves only points in A and $f_1 \in \mathscr{L}_A$ ($\mathscr{L}_A$ is a left ideal), we conclude $f_2 = hf_1$. For each $x \in X \setminus f_1^{-1}(A)$ we have:

$$f_1(x) = hf_1(x) = f_2(x),$$

so $\mathscr{D}_{f_1, f_2} \subseteq f_1^{-1}(A)$. Now if $x \in f_1^{-1}(A)$, that is $f_1(x) = a_i$, $i = 1, 2$, then

$$f_1(x) = a_i \neq h(a_i) = hf_1(x) = f_2(x),$$

hence $\mathscr{D}_{f_1, f_2} \supseteq f_1^{-1}(A)$. We conclude

$$\mathscr{D}_{f_1, f_2} = f_1^{-1}(A).$$

Thus

$$\begin{aligned} D_{f_1, f_2} &= \{\{f_1(x), f_2(x)\} : x \in \mathscr{D}_{f_1, f_2}\} \quad \text{(Notation 2.6)} \\ &= \{\{f_1(x), f_2(x)\} : x \in f_1^{-1}(A)\} \\ &= \{\{a_i, h(a_i)\} : \ i = 1, 2\} \\ &= \{\{a_1, a_2\}\} \\ &= \{A\}, \end{aligned}$$

as required. □

Result 3.11. *Given distinct f_1 and f_2 in S, $\mathscr{L}_{f_1, f_2}$ is a maximal function left ideal if and only if $|D_{f_1, f_2}| = 1$.*

Proof. Let $\mathscr{L}_{f_1, f_2}$ be a maximal function left ideal and suppose $A, B \in D_{f_1, f_2}$, $A \neq B$.

Then $A, B \in \mathscr{P}_2$ (Remark 3.8). Hence

$$\begin{aligned}\mathscr{L}_{f_1,f_2} &= \bigcap_{C \in D_{f_1,f_2}} \mathscr{L}_C \quad \text{(Result 3.9)}\\ &\subseteqq \mathscr{L}_A \cap \mathscr{L}_B\\ &\subsetneqq \mathscr{L}_A \qquad \text{(Lemma 3.4)}\\ &= \mathscr{L}_{g_1,g_2} \qquad \text{(Result 3.10)},\end{aligned}$$

for some distinct $g_1, g_2 \in S$, contradicting the maximality of $\mathscr{L}_{f_1,f_2}$. Hence $|D_{f_1,f_2}| = 1$.

Conversely, suppose $D_{f_1,f_2} = \{A\}$, some $A \in \mathscr{P}_2$, while there exists a function left ideal $\mathscr{L}_{g_1,g_2}$ ($g_1, g_2 \in S$) such that

$$\mathscr{L}_{g_1,g_2} \supseteqq \mathscr{L}_{f_1,f_2}.$$

Since $\mathscr{L}_{g_1,g_2} = \bigcap_{B \in D_{g_1,g_2}} \mathscr{L}_B$ (Result 3.9) we have

$$\bigcap_{B \in D_{g_1,g_2}} \mathscr{L}_B = \mathscr{L}_{g_1,g_2} \supseteqq \mathscr{L}_{f_1,f_2} = \mathscr{L}_A \quad \text{(Result 3.9 again)},$$

and so Lemma 3.4 ensures $D_{g_1,g_2} = \{A\}$, that is

$$\mathscr{L}_{g_1,g_2} = \mathscr{L}_A = \mathscr{L}_{f_1,f_2}.$$

□

Corollary 3.12. *Given f_1 and f_2 is S, $\mathscr{L}_{f_1,f_2}$ is a maximal left function ideal if and only if $\mathscr{L}_{f_1,f_2} = \mathscr{L}_A$, some $A \in \mathscr{P}_2$.*

Proof. Follows from Results 3.9 and 3.11. □

We show now that each automorphism ϕ of S permutes point left ideals.

Result 3.13. *Given $A \in \mathscr{P}_2$,*

$$\phi(\mathscr{L}_A) = \mathscr{L}_B,$$

for some $B \in \mathscr{P}_2$.

Proof. Choose f_1 and f_2 in S such that $\mathscr{L}_{f_1,f_2}=\mathscr{L}_A$ (Result 3.10), then

$$\begin{aligned}\phi(\mathscr{L}_A)=\phi(\mathscr{L}_{f_1,f_2})&=\phi(\{l: lf_1=lf_2\})\\&=\{\phi(l): \phi(lf_1)=\phi(lf_2)\}\\&=\{\phi(l): \phi(l)\phi(f_1)=\phi(l)\phi(f_2)\}\\&=\{l': l'\phi(f_1)=l'\phi(f_2)\}\\&=\mathscr{L}_{\phi(f_1),\phi(f_2)}.\end{aligned}$$

Now Corollary 3.12 ensures $\mathscr{L}_{f_1,f_2}$ is a maximal function left ideal, hence $\mathscr{L}_{\phi(f_1),\phi(f_2)}$ $(=\phi(\mathscr{L}_{f_1,f_2}))$ is a maximal function left ideal, so there exists $B\in\mathscr{P}_2$ such that

$$\mathscr{L}_{\phi(f_1),\phi(f_2)}=\mathscr{L}_B \quad \text{(Corollary 3.12).}$$

We conclude

$$\phi(\mathscr{L}_A)=\mathscr{L}_{\phi(f_1),\phi(f_2)}=\mathscr{L}_B. \qquad \square$$

Define a map

$$\mu: \{\mathscr{L}_A: A\in\mathscr{P}_2\}\to\{\mathscr{L}_A: A\in\mathscr{P}_2\}$$

via $\mu(\mathscr{L}_A)=\phi(\mathscr{L}_A)$, each $\mathscr{L}_A\subseteq S$.

Lemma 3.14. *μ is a bijection.*

Proof. That μ is a mapping is the content of Result 3.13. Similarly by considering the automorphism ϕ^{-1} we define a map

$$\xi: \{\mathscr{L}_A: A\in\mathscr{P}_2\}\to\{\mathscr{L}_A: A\in\mathscr{P}_2\}$$

via $\xi(\mathscr{L}_A)=\phi^{-1}(\mathscr{L}_A)$, each $\mathscr{L}_A\subseteq S$. Certainly, ξ is the inverse of μ and so μ is a bijection. $\square$

We now define a map

$$\lambda: \mathscr{P}_2\to\mathscr{P}_2 \quad \text{via} \quad \lambda(A)=B, \quad \text{where} \quad \mu(\mathscr{L}_A)=\mathscr{L}_B, \quad \text{each} \quad A\in\mathscr{P}_2.$$

It is clear that

$$\lambda=\delta^{-1}\mu\delta,$$

and so Lemmas 3.5 and 3.14 ensure λ is a bijection of $\mathscr{P}_2$. We call λ the *bijection of $\mathscr{P}_2$ associated with ϕ.*

We show that λ is *induced* by a bijection h of X, that is

$$\lambda(A) = h(A),$$

for each $A \in \mathscr{P}_2$. Note here that not every bijection of $\mathscr{P}_2$ is induced, as shown in Example 3.15 below.

Example 3.15. Fix A and C in $\mathscr{P}_2$, $A \neq C$ and let λ be a bijection of $\mathscr{P}_2$, which interchanges A and C and the identity otherwise. Choose $B \in \mathscr{P}_2$, $B = \{x, y\}$ such that $x \in A \backslash C$ and $y \in X \backslash (A \cup C)$. Note $A \cap B = \{x\}$ and $B \cap C = \Phi$. Suppose λ is induced by $h \in \mathscr{G}_X$, then

$$h(x) = h(A \cap B) = h(A) \cap h(B) = \lambda(A) \cap \lambda(B) = C \cap B = \Phi.$$

Thus λ is not induced. $\square$

Observe that in the example above we had λ, a bijection of $\mathscr{P}_2$, such that

$$|A \cap B| \neq |\lambda(A) \cap \lambda(B)|,$$

for some A, B in $\mathscr{P}_2$. This leads us to a criterion for a bijection λ of $\mathscr{P}_2$ to be induced.

Result 3.16. *Let λ be a bijection of $\mathscr{P}_2$. Then λ is induced if and only if $|A \cap B| = |\lambda(A) \cap \lambda(B)|$, for every $A, B \in \mathscr{P}_2$.*

Proof. If λ is induced by an $h \in \mathscr{G}_X$, then for every $A, B \in \mathscr{P}_2$, $|A \cap B| = |h(A \cap B)| = |h(A) \cap h(B)| = |\lambda(A) \cap \lambda(B)|$.

For the converse suppose that λ is a bijection of $\mathscr{P}_2$ such that for every $A, B \in \mathscr{P}_2$

$$|A \cap B| = |\lambda(A) \cap \lambda(B)|. \tag{*}$$

We show that λ is induced. This is done in the following three steps.

1. *Given $x \in X$ there exists a unique $y \in X$ such that for every $A, B \in \mathscr{P}_2$ with $A \cap B = \{x\}$ we have $\lambda(A) \cap \lambda(B) = \{y\}$.*

Take a pair A, B in $\mathscr{P}_2$ with $A \cap B = \{x\}$, then by the assumption (*) $\lambda(A) \cap \lambda(B) = \{y\}$, for some $y \in X$.

Take any other pair C, D in $\mathscr{P}_2$ with $|C \cap D| = 1$ and let $\mathscr{F} \subseteq \mathscr{P}_2$ be such that:

(α) for every distinct $F_1, F_2 \in \mathscr{F}, |F_1 \cap F_2| = 1$;

(β) for any $F \in \mathscr{F}, |A \cap F| = |B \cap F| = |C \cap F| = |D \cap F| = 1$.

We show:

$$C \cap D = \{x\} \text{ iff there exists an } \mathscr{F} \text{ (as described above) with } |\mathscr{F}| = |X|.$$

Let $A \cup B \cup C \cup D = E$, then $|E| \leqq 8$ and $|X \backslash E| = |X|$.

Assume firstly that $C \cap D = \{x\}$ and let $\mathscr{F} = \{\{x, y\} : y \in X \backslash E\}$. Then $\mathscr{F}$ satisfies (α) and (β) and $|\mathscr{F}| = |X \backslash E| = |X|$.

For the converse assume $C \cap D = \{z\}$, $z \neq x$ and $\mathscr{F} \subseteq \mathscr{P}_2$ satisfies (α) and (β). For each $F \in \mathscr{F}$ we have $|E \cap F| > 1$. (If not, then

$$|E \cap F| = |(A \cup B \cup C \cup D) \cap F|$$
$$= |(A \cap F) \cup (B \cap F) \cup (C \cap F) \cup (D \cap F)| \leqq 1.$$

Using condition (β) we conclude:

$$A \cap F = B \cap F = C \cap F = D \cap F = A \cap B = \{x\},$$

or $C \cap D = \{x\}$, a contradiction).

Define a map $v : \mathscr{F} \to \mathscr{P}(E)$, where $\mathscr{P}(E)$ is the power set of E, via $v(F) = E \cap F$, each $F \in \mathscr{F}$. We show v is 1–1. Suppose $F_1, F_2 \in \mathscr{F}$ with $v(F_1) = v(F_2)$. Then

$$1 < |E \cap F_1| = |E \cap F_1 \cap F_2| \leqq |F_1 \cap F_2|,$$

so that $|F_1 \cap F_2| > 1$, thus $F_1 = F_2$ (condition (α)). However $\mathscr{P}(E)$ is finite, so $|\mathscr{F}| \leqq |\mathscr{P}(E)| < \aleph_0$, or $|\mathscr{F}| < |X|$. We conclude $C \cap D = \{x\}$.

Observe now that the definition of the set $\mathscr{F}$ depends on the sets A, B, C and D. We denote this dependence by $\mathscr{F} = \mathscr{F}(A, B, C, D)$. Hence $C \cap D = \{x\}$ iff $\exists\, \mathscr{F}(A, B, C, D)$ with $|\mathscr{F}(A, B, C, D)| = |X|$ iff $\exists\, \mathscr{F}(\lambda(A), \lambda(B), \lambda(C), \lambda(D))$ with $|\mathscr{F}(\lambda(A), \lambda(B), \lambda(C), \lambda(D))| = |X|$ (assumption (*))

$$\text{iff } \lambda(C) \cap \lambda(D) = \{y\}.$$

Now define a map

$$h : X \to X \text{ via } \{h(x)\} = \lambda(A) \cap \lambda(B), \text{ where } \{x\} = A \cap B, \text{ for } A, B \in \mathscr{P}_2 \text{ and each } x \in X.$$

2. *h is a bijection of X.*

That h is well-defined is the content of step 1. Observe that the bijection λ^{-1} of $\mathscr{P}_2$ is associated with the automorphism ϕ^{-1}. By considering ϕ^{-1} and λ^{-1} instead of ϕ and λ we define a map $k : X \to X$ via $\{k(x)\} = \lambda^{-1}(A) \cap \lambda^{-1}(B)$, where $\{x\} = A \cap B$, for $A, B \in \mathscr{P}_2$ and each $x \in X$. Then for each $x \in X$

$$\{kh(x)\} = k(\lambda(A) \cap \lambda(B)), \quad \text{where} \quad A \cap B = \{x\}$$
$$= \lambda^{-1}\lambda(A) \cap \lambda^{-1}\lambda(B)$$
$$= A \cap B$$
$$= \{x\}.$$

Similarly we can show $hk(x) = x$, for each $x \in X$. Thus k is the inverse of h, and so h is a bijection of X.

3. *λ is induced by h.*

To show λ is induced by h we must show $\lambda(A)=h(A)$ for each $A\in\mathscr{P}_2$. From the definition of h we at once have $h(A)\subseteq\lambda(A)$. Take $y\in\lambda(A)$ and let $B\in\mathscr{P}_2$ be such that $\lambda(A)\cap\lambda(B)=\{y\}$. Then $A\cap B=\{x\}$, some $x\in A$, so $h(x)=y$ and $h(A)\supseteq\lambda(A)$. The equality follows. □

Remark 3.17. In view of Result 3.16 our aim now is to show that for every $A,B\in\mathscr{P}_2$

$$|A\cap B|=|\lambda(A)\cap\lambda(B)| \tag{*}$$

where λ is the bijection of $\mathscr{P}_2$ associated with ϕ as defined prior to Example 3.15. Observe that (*) is equivalent to the statement

$$|A\cap B|=1 \quad\text{if and only if}\quad |\lambda(A)\cap\lambda(B)|=1, \tag{**}$$

for each $A,B\in\mathscr{P}_2$.

Indeed (*) certainly implies (**). We show the reverse implication.

Assume (**) holds. If $|A\cap B|=2$, that is $A=B$, then $\lambda(A)=\lambda(B)$, and so $|\lambda(A)\cap\lambda(B)|=2$. If $|A\cap B|=1$, then by our assumption $|\lambda(A)\cap\lambda(B)|=1$. The case $|A\cap B|=0$ follows by elimination. □

The next lemma illustrates the fact that the existence of a transformation f in S which is not 1–1 provides an extensive variety of elements in $\pi(S)$.

Lemma 3.18. *Given $B_1,B_2\subseteq X$ with $B_1\cap B_2=\Phi$ and $|B_1|=|B_2|=3$ there exists an $\mathscr{A}\in\pi(S)$ with $B_1\subseteq A_1\in\mathscr{A}$, $B_2\subseteq A_2\in\mathscr{A}$.*

Proof. Suppose that there exists a transformation f in S such that:

$$C_1,C_2\in\pi(f) \quad\text{and}\quad |C_1|,|C_2|\geqq 3.$$

Choose a bijection p of X with

$$B_1\subseteq p(C_1) \quad\text{and}\quad B_2\subseteq p(C_2).$$

Certainly $pfp^{-1}\in S$. Let

$$\mathscr{A}=\pi(pfp^{-1})(=p(\pi(f))), A_1=p(C_1) \quad\text{and}\quad A_2=p(C_2),$$

then $A_1,A_2\in\mathscr{A}\in\pi(S)$ and $B_1\subseteq A_1$, $B_2\subseteq A_2$.

To construct such an f as used above we first show that there exists a g in S such that

$$g(x_1)=g(x_2)=g(x_3)=x_1, \quad\text{for some distinct } x_1,x_2,x_3\in X.$$

Choose a t in S not 1–1 and let $x, x_1, x_2 \in X$ be such that

$$t(x_1) = t(x_2) = x.$$

We assume $x = x_1$ (for if $x \neq x_1$ choose $s \in S$ such that $s(x) = x_1$ (Result 1.3) and replace t by st). Let $x_4 \in R(t) \backslash \{t^{-1}(x_1)\}$ (note: $R(t) \backslash \{t^{-1}(x_1)\} \neq \Phi$, else t^2 is a constant in S) and let $x_3 \in X$ such that $t(x_3) = x_4$. Then

$$g = (x_2, x_4)t(x_2, x_4)t$$

is such that $g(x_1) = g(x_2) = g(x_3) = x_1$.

To accomplish the construction of the above f choose distinct z_1, z_2, z_3 in $R(g) \backslash \{g^{-1}(x_1)\}$ together with $y_1, y_2, y_3 \in X$ such that $g(y_i) = z_i$, $i = 1, 2, 3$. Let

$$k = (x_1, z_1)(x_2, z_2)(x_3, z_3) \in \mathscr{G}_X \quad \text{and} \quad f = kgk^{-1}g.$$

Let $kg(z_1) = z_4$. Then

$$f(x_1) = f(x_2) = f(x_3) = z_4$$

and

$$f(y_1) = f(y_2) = f(y_3) = z_1.$$

Now $z_1 \neq z_4$ (else $kg(z_1) = z_1$ implies $g(z_1) = x_1$ or $z_1 \in g^{-1}(x_1)$, contrary to the choice of z_1). Let $C_1 = f^{-1}(z_1)$, $C_2 = f^{-1}(z_4)$. Then $|C_1|, |C_2| \geqq 3$ and $C_1, C_2 \in \pi(f)$ as required. □

Remark 3.19. It easily follows from Lemma 3.18 that

$$\mathscr{L}_A \cap \mathscr{L}_B \neq \Phi,$$

for every $A, B \in \mathscr{P}_2$. □

Lemma 3.20. *Let $A, B \in \mathscr{P}_2$, $A \neq B$. Then $|A \cap B| = 1$ iff there is a C in $\mathscr{P}_2$, $C \neq A$ or B, such that $\mathscr{L}_A \cap \mathscr{L}_B \subseteq \mathscr{L}_C$.*

Proof. Assume $|A \cap B| = 1$ and let $C = (A \cup B) \backslash (A \cap B)$. For each l in $\mathscr{L}_A \cap \mathscr{L}_B$ (Remark 3.19):

$$l(A) = l(A \cap B) = l(B) = l(A \cup B) = l(C),$$

so that $l \in \mathscr{L}_C$ and $\mathscr{L}_A \cap \mathscr{L}_B \subseteq \mathscr{L}_C$.

For the converse suppose $A \cap B = \Phi$ and $C \in \mathscr{P}_2$ is distinct from A and B. Let $C = \{c_1, c_2\}$. Since $|A \cap C| \leqq 1$ and $|B \cap C| \leqq 1$ assume without loss of generality that $c_1 \in X \backslash B$ and $c_2 \in X \backslash A$. Choose

$$\mathscr{A} \in \pi(S) \quad \text{with} \quad A \cup \{c_1\} \subseteq A_1 \in \mathscr{A}, B \cup \{c_2\} \subseteq A_2 \in \mathscr{A} \quad \text{and} \quad A_1 \neq A_2 \quad \text{(Lemma 3.18).}$$

If $l \in S$ has $\pi(l)=\mathscr{A}$, then $l \in (\mathscr{L}_A \cap \mathscr{L}_B)\backslash\mathscr{L}_C$.
This confirms that $|A \cap B|=1$. □

Lemma 3.21. *Let A, B and C be distinct elements of $\mathscr{P}_2$. Then*

$$\mathscr{L}_A \cap \mathscr{L}_B \subseteq \mathscr{L}_C \quad \textit{iff} \quad \mathscr{L}_{\lambda(A)} \cap \mathscr{L}_{\lambda(B)} \subseteq \mathscr{L}_{\lambda(C)}.$$

Proof. Observe that $\mathscr{L}_A \cap \mathscr{L}_B \neq \Phi$ (Remark 3.19) and

$$\mathscr{L}_A \cap \mathscr{L}_B \subseteq \mathscr{L}_C \quad \text{iff} \quad \phi(\mathscr{L}_A \cap \mathscr{L}_B) \subseteq \phi(\mathscr{L}_C).$$

Now

$$\phi(\mathscr{L}_A \cap \mathscr{L}_B) = \phi(\mathscr{L}_A) \cap \phi(\mathscr{L}_B) = \mathscr{L}_{\lambda(A)} \cap \mathscr{L}_{\lambda(B)},$$

by the definition of λ. Also $\phi(\mathscr{L}_C) = \mathscr{L}_{\lambda(C)}$, so that

$$\phi(\mathscr{L}_A \cap \mathscr{L}_B) \subseteq \phi(\mathscr{L}_C) \quad \text{iff} \quad \mathscr{L}_{\lambda(A)} \cap \mathscr{L}_{\lambda(B)} \subseteq \mathscr{L}_{\lambda(C)},$$

and the desired equivalence is established. □

Result 3.22. *Given A and B in $\mathscr{P}_2$,*

$$|A \cap B|=1 \quad \textit{if and only if} \quad |\lambda(A) \cap \lambda(B)|=1.$$

Proof. We have:
$|A \cap B|=1$ iff $\exists\, C \neq A$ or B such that $\mathscr{L}_A \cap \mathscr{L}_B \subseteq \mathscr{L}_C$ (Lemma 3.20)
iff $\exists\, \lambda(C) \neq \lambda(A)$ or $\lambda(B)$ such that $\mathscr{L}_{\lambda(A)} \cap \mathscr{L}_{\lambda(B)} \subseteq \mathscr{L}_{\lambda(C)}$
(λ is a bijection and Lemma 3.21)

iff $|\lambda(A) \cap \lambda(B)|=1$ (Lemma 3.20 again). □

From Results 3.16, 3.22 and Remark 3.17 we readily deduce

Result 3.23. *λ is induced by a bijection of X.* □

Now we are ready to show that a constant-free $\mathscr{G}_X$-normal semigroup containing a transformation which is not 1–1 (that is a semigroup of Type 3), possesses only inner automorphisms.

Result 3.24. *Let S be a constant-free $\mathscr{G}_X$-normal semigroup containing a transformation which is not 1–1. Then each automorphism ϕ of S is inner, that is for some $h \in \mathscr{G}_X$*

$$\phi(f) = hfh^{-1}, \quad \textit{for each} \quad f \in S.$$

Proof. Let h be the bijection which induces λ (Result 3.23). In what follows we use the fact that for any distinct $x_1, x_2 \in X$

$$\phi(\mathscr{L}_{\{x_1,x_2\}}) = \mathscr{L}_{\lambda(\{x_1,x_2\})} = \mathscr{L}_{\{h(x_1),h(x_2)\}}.$$

Take an arbitrary $f \in S$, $x \in X$ and let $y \in X$ with $f(x) \neq f(y)$ (that is $f \notin \mathscr{L}_{\{x,y\}}$). Then

$$\phi(\mathscr{L}_{\{f(x),f(y)\}}) = \mathscr{L}_{\{hf(x),hf(y)\}}.$$

Let $\phi(g) \in \phi(\mathscr{L}_{\{f(x),f(y)\}})$. Then $g \in \mathscr{L}_{\{f(x),f(y)\}}$ or $gf(x) = gf(y)$. It follows that $\phi(gf) \in \mathscr{L}_{\{h(x),h(y)\}}$, hence

$$\phi(g)\phi(f)h(x) = \phi(g)\phi(f)h(y).$$

Note that $f \notin \mathscr{L}_{\{x,y\}}$ implies $\phi(f) \notin \phi(\mathscr{L}_{\{x,y\}})$ or $\phi(f) \notin \mathscr{L}_{\{h(x),h(y)\}}$, that is

$$\phi(f)h(x) \neq \phi(f)h(y).$$

Thus $\phi(g) \in \mathscr{L}_{\{\phi(f)h(x),\phi(f)h(y)\}}$ and we conclude

$$\phi(\mathscr{L}_{\{f(x),f(y)\}}) \subseteq \mathscr{L}_{\{\phi(f)h(x),\phi(f)h(y)\}}.$$

This in turn implies

$$\mathscr{L}_{\{hf(x),hf(y)\}} \subseteq \mathscr{L}_{\{\phi(f)h(x),\phi(f)h(y)\}}.$$

Hence $\{hf(x), hf(y)\} = \{\phi(f)h(x), \phi(f)h(y)\}$ (Lemma 3.4).

Since the choice of y is independent of x (providing $y \neq x$) we conclude

$$\phi(f)h(x) = hf(x), \quad \text{for each} \quad x \in X,$$

so that

$$\phi(f) = hfh^{-1}, \quad \text{as required.}$$ □

Conclusion

We return to

Theorem 1.1. *Every automorphism of a $\mathscr{G}_X$-normal semigroup S is inner.*

Proof. If S is a semigroup of Type 1, that is, contains a constant transformation, then we appeal to Sullivan [**12,** Theorem 1].

If S is a semigroup of Type 2, that is, a semigroup of 1–1 transformations, the result is given in 2.14 and 1.4.

If S is a semigroup of Type 3, that is, a semigroup containing a transformation which is not 1–1, then the result is given in 3.24.

This completes the proof of Theorem 1.1. □

Remark. If X is a *finite* set and S is a semigroup of transformations of X which is not contained in $\mathcal{G}_X$, then S is $\mathcal{G}_X$-normal if and only if all automorphisms of S are inner [13].

However, this is not the case for an infinite set X. While, as we showed, every $\mathcal{G}_X$-normal semigroup S has only inner automorphisms, there are examples [5] of semigroups which are neither subsemigroups of $\mathcal{G}_X$, nor $\mathcal{G}_X$-normal, yet have only inner automorphisms.

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