AUTOMORPHISMS OF NORMAL TRANSFORMATION SEMIGROUPS

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1. Introduction and preliminaries

Let \( X \) be an infinite set, \( \mathcal{G}_X \) be the group of all bijections of \( X \) and \( S \) be a semigroup of total transformations of \( X \) with the composition of transformations \( f \) and \( g \) in \( S \) defined by the formula

\[
fg(x) = f(g(x)), \quad \text{where} \quad x \in X.
\]

We say that \( S \) is a \( \mathcal{G}_X \)-normal semigroup if

\[
hSh^{-1} = S, \quad \text{for all} \quad h \in \mathcal{G}_X.
\]

The full transformation semigroup \( T_X \), the semigroups of all 1–1 and all onto transformations and the group \( \mathcal{G}_X \) itself, are examples of \( \mathcal{G}_X \)-normal semigroups.

If \( S \) is a \( \mathcal{G}_X \)-normal semigroup, then for each \( h \in \mathcal{G}_X \), the map \( \phi \) of \( S \) given by

\[
\phi(f) = hfh^{-1} \quad (f \in S)
\]

is an automorphism of \( S \), specifically an inner automorphism of \( S \). Our purpose is to prove the following:

**Theorem 1.1.** Every automorphism of a \( \mathcal{G}_X \)-normal semigroup is inner.

The subject of this paper was suggested to the author by G. R. Wood.

The question of whether inner automorphisms exhaust all automorphisms of a \( \mathcal{G}_X \)-normal semigroup has attracted the attention of a number of authors. In 1937 Schreier [10] was the first to give a positive answer for \( T_X \). Then Malcev [6] extended this result to every ideal of \( T_X \). Next Sullivan [12] generalized this work and confirmed that if a semigroup contains all constant transformations (in particular if a \( \mathcal{G}_X \)-normal semigroup contains a constant transformation) then it possesses only inner automorphisms, while Fitzpatrick and Symons [3] showed this for a semigroup containing \( \mathcal{G}_X \). Schein [8, 9] discovered that a \( \mathcal{G}_X \)-normal semigroup of 1–1 transformations has only inner automorphisms (see [4] for the special case of Baer–Levi semigroups).

Our result subsumes all previously stated results for \( \mathcal{G}_X \)-normal semigroups and describes completely all automorphisms of every \( \mathcal{G}_X \)-normal transformation semigroup.
In this paper we use a technique which differs from those used by Sullivan [12] and Schein [8,9]. The essence is the production of certain maximal right (Section 2) and left (Section 3) ideals. We note a remarkable duality between properties of these right and left ideals.

For the purpose of our proof we partition all $S_X$-normal semigroups into three types:
1. Semigroups containing a constant map; and constant-free semigroups into:
   2. Semigroups of 1-1 transformations; and
   3. Constant-free semigroups containing a transformation which is not 1-1.

All automorphisms of semigroups of the first type are inner [12, Theorem 1], so we can restrict our attention to constant-free semigroups.

We begin with some general notes on $S_X$-normal semigroups.

For a function $f : X \to X$ we denote the range of $f$ by $R(f) (= f(X))$ and the partition of $f$ by $\pi(f) (= \{f^{-1}(x) : x \in R(f)\})$.

If $S$ is an arbitrary semigroup of transformations, let

$$R(S) = \{R(f) : f \in S\} \quad \text{and} \quad \pi(S) = \{\pi(f) : f \in S\}.$$ 

We say that $R(S)$ $(\pi(S))$ is normal if for each $h \in S_X$

$$h(R(S)) = R(S) \quad (h(\pi(S)) = \pi(S)),$$

(by $h(R(S))$ we mean $\{h(A) : A \in R(S)\}$ and by $h(\pi(S))$ we mean $\{h(A) : A \in \pi(S)\}$, where $h(A) = \{h(a) : a \in A\}$).

**Lemma 1.2.** If $S$ is a $S_X$-normal semigroup, then $R(S)$ and $\pi(S)$ are normal.

The proof is straightforward. \hfill \Box

We say that a semigroup $S$ is trivial if $S = \{\Delta_X\}$, where $\Delta_X$ is the identity transformation of $X$. In what follows $S$ is non-trivial.

**Result 1.3.** Every $S_X$-normal semigroup $S$ is transitive.

**Proof.** Take arbitrary $x, y$ in $X$. We construct $f$ in $S$ such that $f(x) = y$.

Firstly let $x$ and $y$ be distinct and suppose there exists a $g \in S$ with $g(x) = z \neq x$. If $z = y$ we let $f = g$, otherwise $(y,z)g(y,z)$ is the required $f$ ($(y,z)$ denotes the transposition interchanging $y$ and $z$). To construct $g$, observe that since $S$ is non-trivial there exists a $q \in S$ together with distinct $u$ and $v$ in $X$ such that $q(u) = v$. If $u = x$ we let $g = q$, otherwise $g = (u,x)q(u,x)$.

Now suppose $y = x$, choose any $p$ in $S$ and let $p(x) = w$. If $w = x$ we let $f = p$. Otherwise choose $t \in S$ with $t(w) = x$ (using the first part of the proof), then $f = tp$ takes $x$ to $x$ as required. \hfill \Box

**Remark 1.4.** We exclude from our consideration $S_X$-normal subsemigroups of $S_X$, since they are all subgroups of $S_X$, and hence have only inner automorphisms [11].
2. $\mathcal{G}_x$-normal semigroups of 1-1 transformations

In this section $S$ denotes a $\mathcal{G}_x$-normal semigroup of 1-1 transformations.

**Definition 2.1.** Let $x \in X$ and

$$\mathcal{R}_x = \{ r \in S : x \in X \setminus R(r) \}.$$ 

Then $\mathcal{R}_x$ is a right ideal of $S$, which we call a point right ideal. 

We will use the following observation based on the normality of $R(S)$ (Lemma 1.2) and the fact that $S$ is not a subsemigroup of $\mathcal{G}_x$, that is $R(S)$ contains proper subsets of $X$.

**Remark 2.2.** Given $x, y \in X$ with $x \neq y$ there exists an $A$ in $R(S)$ with $x \in X \setminus A$ and $y \in A$. 

**Lemma 2.3.** Given $x, y \in X$ the following three statements are equivalent:

(i) $x = y$;

(ii) $x \neq y$.

(iii) $\mathcal{R}_x = \mathcal{R}_y$.

**Proof.** Implications (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(i) are trivial. We show (i)$\Rightarrow$(ii). Suppose $x \neq y$ and choose an $A \in R(S)$ with $x \in X \setminus A$, $y \in A$ (Remark 2.2). If $f \in S$ with $R(f) = A$, then $f \in \mathcal{R}_x \setminus \mathcal{R}_y$, so $\mathcal{R}_x \neq \mathcal{R}_y$, proving (i)$\Rightarrow$(ii). 

Define a map $\theta : X \to \{ \mathcal{R}_x : x \in X \}$ via $\theta(x) = \mathcal{R}_x$, each $x \in X$.

**Lemma 2.4.** $\theta$ is a bijection.

**Proof.** Clearly $\theta$ is onto and Lemma 2.3 ensures $\theta$ is 1-1.

**Definition 2.5.** Given distinct $f_1, f_2 \in S$ let

$$\mathcal{R}_{f_1, f_2} = \{ r \in S : f_1 r = f_2 r \}.$$ 

Then $\mathcal{R}_{f_1, f_2}$ is a right ideal of $S$ (possibly empty), which we call a function right ideal.

We will show (Result 2.8) that there always exist distinct $f_1, f_2$ in $S$ such that $\mathcal{R}_{f_1, f_2}$ is non-empty. However $\mathcal{R}_{f_1, f_2}$ may be empty. Observe that given $f_1$ and $f_2$,

$$r \in \mathcal{R}_{f_1, f_2} \iff R(r) \subseteq \{ x \in X : f_1(x) = f_2(x) \}.$$ 

Hence if we choose $f_1$ and $f_2$ which are never equal, then $\mathcal{R}_{f_1, f_2} = \emptyset$.

Let $S$, for example, be the Baer–Levi semigroup of type $(|X|, |X|)$ [2], that is the semigroup of all 1-1 transformations $f$ such that $|R(f)| = |X \setminus R(f)| = |X|$. Note that $S$ is
\(\mathcal{D}^{X}\)-normal and choose \(f_1 \in S\), then \(X \setminus R(f_1) \subseteq R(S)\) (Lemma 1.2). If \(f_2 \in S\) with \(R(f_2) = X \setminus R(f_1)\), then \(\mathcal{R}_{f_1, f_2} = \emptyset\).

The following notation applies to an arbitrary \(\mathcal{D}^{X}\)-normal semigroup \(S\).

**Notation 2.6.** Let \(f_1, f_2\) be distinct transformations in \(S\). Then

\[
\mathcal{D}_{f_1, f_2} = \{ x \in X : f_1(x) \neq f_2(x) \}
\]

and

\[
D_{f_1, f_2} = \{ \{ f_1(x), f_2(x) \} : x \in \mathcal{D}_{f_1, f_2} \}.
\]

Returning to semigroups of 1-1 transformations, we now derive relationships between point right ideals and function right ideals.

**Result 2.7.** Let \(f_1, f_2 \in S\) with \(\mathcal{R}_{f_1, f_2} \neq \emptyset\). Then

\[
\mathcal{R}_{f_1, f_2} = \bigcap_{x \in \mathcal{D}_{f_1, f_2}} \mathcal{R}_x.
\]

**Proof.** Let \(r \in \mathcal{R}_{f_1, f_2}\), that is \(f_1 r = f_2 r\). If \(x \in \mathcal{D}_{f_1, f_2}\), or \(f_1(x) \neq f_2(x)\), then \(x \in X \setminus R(r)\), so \(r \in \mathcal{R}_x\), and since this is true for each \(x \in \mathcal{D}_{f_1, f_2}\), we conclude

\[
r \in \bigcap_{x \in \mathcal{D}_{f_1, f_2}} \mathcal{R}_x,
\]

or

\[
\mathcal{R}_{f_1, f_2} \subseteq \bigcap_{x \in \mathcal{D}_{f_1, f_2}} \mathcal{R}_x.
\]

Conversely, if

\[
r \in \bigcap_{x \in \mathcal{D}_{f_1, f_2}} \mathcal{R}_x,
\]

then for each \(y\) in \(R(r)\) we have \(y \in X \setminus \mathcal{D}_{f_1, f_2}\), or \(f_1(y) = f_2(y)\) and hence \(f_1 r = f_2 r\), that is \(r \in \mathcal{R}_{f_1, f_2}\), so

\[
\bigcap_{x \in \mathcal{D}_{f_1, f_2}} \mathcal{R}_x \subseteq \mathcal{R}_{f_1, f_2},
\]

which proves the desired equality. \(\square\)

**Result 2.8.** Given \(x \in X\) there exist \(f_1, f_2 \in S\) such that \(\mathcal{R}_x = \mathcal{R}_{f_1, f_2}\).

**Proof.** On account of Result 2.7 it is sufficient to construct \(f_1, f_2\) such that \(\mathcal{D}_{f_1, f_2} = \{x\}\).

Observe that there exists an \(f\) in \(S\) with

\[
|X \setminus R(f)| \geq 2.
\]
(For an arbitrary \( f \) in \( S \backslash \mathcal{G}_X \)

\[ |X \setminus R(f^2)| = |X \setminus R(f)| + |X \setminus R(f)| \]

and we replace \( f \) with \( f^2 \).)

Using the normality of \( R(S) \) (Lemma 1.2) choose an \( f \) in \( S \) with

\[ x \in X \setminus R(f) \quad \text{and} \quad |X \setminus R(f)| \geq 2. \]

Let \( f(x) = y \) and \( z \in X \setminus R(f) \), \( z \neq x \). If

\[ g = (x, z) f(x, z) \]

then \( g(z) = y \) and \( z \in X \setminus R(g) \). We let

\[ h = (y, z), \quad f_1 = g f \quad \text{and} \quad f_2 = h g h^{-1} f. \]

Then for each \( u \neq x \):

\[ f_1(u) = g f(u) = g h^{-1} f(u), \quad \text{since} \quad f(u) \neq y \quad \text{for} \quad u \neq x \]

and \( z \notin R(f) \); 

\[ = h g h^{-1} f(u), \quad \text{since} \quad g h^{-1} f(u) \neq y \]

for \( f(u) \neq y \)

and \( z \notin R(g) \);

\[ = f_2(u). \]

However

\[ f_1(x) = g f(x) = g(y) \]

while

\[ f_2(x) = h g h^{-1} f(x) = h g h^{-1}(y) = h g(x) = h(y) = z \neq g(y), \]

since \( z \in X \setminus R(g) \). Hence \( f_1(x) \neq f_2(x) \) and \( \mathcal{D}_{f_1, f_2} = \{ x \} \).

**Result 2.9.** Given \( f_1 \) and \( f_2 \) in \( S \), \( \mathcal{R}_{f_1, f_2} \) is a maximal function right ideal if and only if \( |\mathcal{D}_{f_1, f_2}| = 1 \).

**Proof.** Suppose \( \mathcal{R}_{f_1, f_2} \) is a maximal function right ideal, while \( x, y \in \mathcal{D}_{f_1, f_2}, \ x \neq y \).
Then

\[ \mathbb{R}_{f_1,f_2} = \bigcap_{z \in \mathbb{R}_{f_1,f_2}} \mathbb{R}_z \quad \text{(Result 2.7)} \]

\[ \subseteq \mathbb{R}_x \cap \mathbb{R}_y \]

\[ \subseteq \mathbb{R}_x \quad \text{(Lemma 2.3)}. \]

It follows from Result 2.8 that there exist \( g_1 \) and \( g_2 \) with

\[ \mathbb{R}_{g_1,g_2} = \mathbb{R}_x, \]

and so

\[ \mathbb{R}_{f_1,f_2} \subseteq \mathbb{R}_x = \mathbb{R}_{g_1,g_2}, \]

a contradiction to the maximality of \( \mathbb{R}_{f_1,f_2} \). Hence \( |\mathbb{R}_{f_1,f_2}| = 1 \).

For the converse, suppose \( \mathbb{R}_{f_1,f_2} = \{x\} \), some \( x \in X \), while there exist \( g_1, g_2 \in S \) such that

\[ \mathbb{R}_{g_1,g_2} \supseteq \mathbb{R}_{f_1,f_2}. \]

Since

\[ \mathbb{R}_{g_1,g_2} = \bigcap_{y \in \mathbb{R}_{g_1,g_2}} \mathbb{R}_y \quad \text{(Result 2.7)} \]

we have

\[ \bigcap_{y \in \mathbb{R}_{g_1,g_2}} \mathbb{R}_y = \mathbb{R}_{g_1,g_2} \supseteq \mathbb{R}_{f_1,f_2} = \mathbb{R}_x \quad \text{(Result 2.7 again)}, \]

and so Lemma 2.3 ensures \( \mathbb{R}_{g_1,g_2} = \{x\} \), that is

\[ \mathbb{R}_{g_1,g_2} = \mathbb{R}_x = \mathbb{R}_{f_1,f_2}. \]

**Corollary 2.10.** Given \( f_1 \) and \( f_2 \) in \( S \), \( \mathbb{R}_{f_1,f_2} \) is a maximal function right ideal if and only if \( \mathbb{R}_{f_1,f_2} = \mathbb{R}_x \), some \( x \in X \).

**Proof.** Follows from Results 2.7 and 2.9.

We show now that each automorphism \( \phi \) of \( S \) permutes point right ideals.

**Result 2.11.** Given \( x \in X \),

\[ \phi(\mathbb{R}_x) = \mathbb{R}_y, \]

for some \( y \in X \).

\[ \square \]
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Proof. Choose \( f_1 \) and \( f_2 \) in \( S \) such that \( \mathcal{R}_{f_1, f_2} = \mathcal{R}_x \) (Result 2.8), then

\[
\phi(\mathcal{R}_x) = \phi(\mathcal{R}_{f_1, f_2}) = \phi(\{r : f_1 r = f_2 r\})
\]

\[
= \{\phi(r) : \phi(f_1 r) = \phi(f_2 r)\}
\]

\[
= \{\phi(r) : \phi(f_1)\phi(r) = \phi(f_2)\phi(r)\}
\]

\[
= \{r' : \phi(f_1)r' = \phi(f_2)r'\}
\]

\[
= \mathcal{R}_{\phi(f_1), \phi(f_2)}.
\]

Now Corollary 2.10 ensures \( \mathcal{R}_{f_1, f_2} \) is a maximal function right ideal, hence \( \mathcal{R}_{\phi(f_1), \phi(f_2)}(= \phi(\mathcal{R}_{f_1, f_2})) \) is a maximal function right ideal, so there exists \( y \in X \) such that

\[
\mathcal{R}_{\phi(f_1), \phi(f_2)} = \mathcal{R}_y \quad \text{(Corollary 2.10)}
\]

and thus

\[
\phi(\mathcal{R}_x) = \mathcal{R}_{\phi(f_1), \phi(f_2)} = \mathcal{R}_y.
\]

Define a map

\[
\eta: \{\mathcal{R}_x : x \in X\} \to \{\mathcal{R}_x : x \in X\}
\]

via \( \eta(\mathcal{R}_x) = \phi(\mathcal{R}_x) \), each \( \mathcal{R}_x \subseteq S \).

Lemma 2.12. \( \eta \) is a bijection.

Proof. That \( \eta \) is a mapping is the content of Result 2.11. Similarly by considering the automorphism \( \phi^{-1} \) we define a map

\[
\zeta: \{\mathcal{R}_x : x \in X\} \to \{\mathcal{R}_x : x \in X\}
\]

via \( \zeta(\mathcal{R}_x) = \phi^{-1}(\mathcal{R}_x) \), each \( \mathcal{R}_x \subseteq S \).

Certainly, \( \zeta \) is the inverse of \( \eta \) and so \( \eta \) is a bijection.

We now define a map

\[
h: X \to X \quad \text{via} \quad h(x) = y, \quad \text{where} \quad \eta(\mathcal{R}_x) = \mathcal{R}_y, \quad \text{each} \ x \in X.
\]

It is clear, that

\[
h = \theta^{-1}\eta\theta,
\]

and so Lemmas 2.4 and 2.12 ensure \( h \) is a bijection of \( X \). We call \( h \) the bijection associated with \( \phi \).
Lemma 2.13. Given \( f \in S \),
\[
R(\phi(f)) = h(R(f)).
\]

Proof. Observe that to show \( R(\phi(f)) = h(R(f)) \) it is sufficient to show that
\[
X \setminus R(\phi(f)) = h(X \setminus R(f)),
\]
because for the bijection \( h \), \( h(X \setminus R(f)) = X \setminus h(R(f)) \).
Now if \( x \in X \setminus R(f) \), that is \( f \in \mathbb{R}_x \), then \( \phi(f) \in \eta(\mathbb{R}_x) = \mathbb{R}_{h(x)} \), so \( h(x) \in X \setminus R(\phi(f)) \), or
\[
h(X \setminus R(f)) \subseteq X \setminus R(\phi(f)).
\]
To show the reverse inclusion is true, observe that \( h^{-1} = \theta^{-1} \eta^{-1} \theta \) is the bijection associated with \( \phi^{-1} \) and so the first part of the proof implies that given \( g \in S \),
\[
h^{-1}(X \setminus R(g)) \subseteq X \setminus R(\phi^{-1}(g)).
\]
In particular taking \( g = \phi(f) \) we have \( h^{-1}(X \setminus R(\phi(f))) \subseteq X \setminus R(\phi^{-1}(\phi(f))) \), or
\[
h(X \setminus R(f)) \supseteq X \setminus R(\phi(f)),
\]
and the equality follows. \( \square \)

We complete our study of automorphisms of \( \mathcal{G}_\mathcal{X} \)-normal semigroups of 1-1 transformations, that is, semigroups of Type 2, by presenting the following result.

Result 2.14. Let \( S \) be a \( \mathcal{G}_\mathcal{X} \)-normal semigroup of 1-1 transformations \((S \subseteq \mathcal{G}_\mathcal{X})\). Then each automorphism \( \phi \) of \( S \) is inner, that is, for some \( h \in \mathcal{G}_\mathcal{X} \)
\[
\phi(f) = h f h^{-1}, \quad \text{for each } f \in S.
\]

Proof. Consider the bijection \( h \) associated with \( \phi \) as defined prior to Lemma 2.13. Take an arbitrary \( f \in S \), \( x \in X \) and let \( f(x) = y \). Choose \( A \) in \( R(S) \) with \( A \neq X \) and \( x \in A \).
Let \( z \in X \setminus A \) and \( B=(A\setminus \{x\}) \cup \{z\} \in R(S) \) (Lemma 1.2). Choose \( p \) and \( q \) in \( S \) such that \( R(p) = A \) and \( R(q) = B \).
Now \( R(p) \setminus R(q) = A \setminus B = \{x\} \), thus \( R(fp) \setminus R(fq) = \{f(x)\} = \{y\} \). Using Lemma 2.13 we have:
\[
R(\phi(p)) \setminus R(\phi(q)) = \{h(x)\}
\]
and
\[
R(\phi(fp)) \setminus R(\phi(fq)) = \{h(y)\}.
\]
However
\[
R(\phi(fp)) \setminus R(\phi(fq)) = R(\phi(\phi(p)) \setminus R(\phi(f)p(q)) \setminus R(\phi(f))\phi(q)) = \{\phi(f)h(x)\},
\]
so

\[ \phi(f)h(x) = h(y) = hf(x), \quad \text{that is} \]
\[ \phi(f) = hf h^{-1}. \]

**Remark 2.15.** The fact that every \( G_X \)-normal semigroup of 1-1 transformations possesses only inner automorphisms was first established by B. M. Schein [8, 9]. We understand that his proof, based on the study of ordered sets of ranges, is quite different from ours.

3. \( G_X \)-normal constant-free semigroups containing a transformation which is not 1-1

Let \( S \) be a \( G_X \)-normal constant-free semigroup containing a transformation which is not 1-1. We prove that all automorphisms of \( S \) are inner. We start by showing that \( R(S) \) contains only sets of cardinality \( |X| \).

**Lemma 3.1.** If \( S \) is a \( G_X \)-normal constant-free semigroup, then \( |R(f)| = |X| \), each \( f \in S \).

**Proof.** Suppose there is an \( f \) in \( S \) with \( |R(f)| = \alpha < |X| \), that is \( |\pi(f)| = |R(f)| = \alpha \). We show that there exists an \( A \in \pi(f) \) with \( |A| \geq \alpha \). The result is clear when \( \alpha \) is finite. Hence assume \( \alpha \) is infinite and denote by \( \alpha^+ \) the cardinal successor of \( \alpha \). Then either \( \alpha^+ = |X| \) (and so \( |X| \) is regular [7, 21.14]) or there exists \( \beta < |X| \), \( \beta = \alpha^+ \) (and so \( \beta \) is regular [7, 21.14]). The assumption that each \( A \in \pi(f) \) has a cardinality less than \( \alpha \) implies that \( \cup \pi(f) \geq |X| \) or \( \cup \pi(f) < \beta < |X| \) respectively [7, 21.18], a contradiction. Hence we can choose an \( A \in \pi(f) \) with \( |A| \geq \alpha \) and a \( B \in R(S) \) with \( B \subseteq A \) and \( |B| = \beta \) (Lemma 1.2) together with a \( g \in S \) such that \( R(g) = B \). Then \( |R(fg)| = 1 \), so that \( fg \) is a constant map in \( S \), a contradiction which proves \( |R(f)| = |X| \).

Let \( \mathcal{P}_2 \) be the set of all doubletons in \( X \).

**Definition 3.2.** Given \( A \in \mathcal{P}_2 \), \( A = \{a_1, a_2\} \), let

\[ \mathcal{L}_A = \{ l \in S : l(a_1) = l(a_2) \} \]

Then \( \mathcal{L}_A \) is a left ideal of \( S \) which we call a point left ideal.

**Lemma 3.3.** For each \( A \in \mathcal{P}_2 \), \( \mathcal{L}_A \neq \emptyset \).

**Proof.** Choose a map \( f \) in \( S \) which is not 1-1, say \( f(x) = f(y) \) for distinct \( x, y \in X \). If \( h \in G_X \) is such that \( \{h(x), h(y)\} = A \) then \( hf h^{-1} \in \mathcal{L}_A \).
Lemma 3.4. Given \( A, B \in \mathcal{P}_2 \), the following three statements are equivalent:

(i) \( \mathcal{L}_A \subseteq \mathcal{L}_B \);
(ii) \( A = B \);
(iii) \( \mathcal{L}_A = \mathcal{L}_B \).

Proof. Implications (ii) \( \Rightarrow \) (iii) and (iii) \( \Rightarrow \) (i) are trivial. We show (i) \( \Rightarrow \) (ii).

Let \( B = \{b_1, b_2\} \) and suppose \( A \neq B \), say \( b_1 \in B \setminus A \). Choose an \( l \in \mathcal{L}_A \) (Lemma 3.3) and let \( x \in R(l) \setminus (A \cup B) \) (note: \( |X| = |R(l)| > |(A \cup B)| \), Lemma 3.1). If \( y \in X \) is such that \( l(y) = x \), let \( h = (b_1, y) \) and \( f = hlh^{-1} \). We show \( f \in \mathcal{L}_A \setminus \mathcal{L}_B \). That \( f \in \mathcal{L}_A \) follows from the fact that \( h \) moves only points \( b_1 \) and \( y \), which are not in \( A \). To show that \( f \notin \mathcal{L}_B \), observe that \( f(b_1) = hlh^{-1}(b_1) = h(y) = h(x) \), while \( f(b_2) = hlh^{-1}(b_2) = h(b_2) \), because \( b_2 \neq y \) (else \( x = l(y) = l(b_2) = l(B) \), contrary to the choice of \( x \)). Hence \( f(b_2) \neq h(x) \), because \( l(b_2) = l(B) \neq x \). Thus \( f(b_1) \neq f(b_2) \) and \( f \notin \mathcal{L}_B \).

Define a map \( \delta: \mathcal{P}_2 \to \{ \mathcal{L}_A; A \in \mathcal{P}_2 \} \) via \( \delta(A) = \mathcal{L}_A \), each \( A \in \mathcal{P}_2 \).

Lemma 3.5. \( \delta \) is a bijection.

Proof. Clearly \( \delta \) is onto and Lemma 3.4 ensures \( \delta \) is 1-1.

Definition 3.6. Given distinct \( f_1, f_2 \in S \) let

\[
\mathcal{L}_{f_1, f_2} = \{ l \in S; lf_1 = lf_2 \}.
\]

Then \( \mathcal{L}_{f_1, f_2} \) is a left ideal of \( S \) (possibly empty, see Example 3.7 below), which we call a function left ideal.

We will show (Result 3.10) that for each \( S \)-normal constant-free semigroup \( S \) containing a transformation which is not 1-1 there exist \( f_1, f_2 \in S \) with \( \mathcal{L}_{f_1, f_2} \neq \Phi \). In general, the question of whether \( f_1, f_2 \in S \) generate a non-empty \( \mathcal{L}_{f_1, f_2} \) is the question of whether the equation \( lf_1 = lf_2 \) has a solution \( l \) in \( S \). The example below illustrates that \( \mathcal{L}_{f_1, f_2} \) may be empty.

Example 3.7. Let \( S \) be the dual Baer-Levi semigroup of the type \((|X|, |X|)[1]\), that is the semigroup of all onto mappings \( f \) such that \( |f^{-1}(x)| = |X| \) for each \( x \in X \). Certainly \( S \) is \( S_X \)-normal. Assume \( X = \mathbb{N} \), so that \( |X| = \aleph_0 \). Fix an arbitrary \( f_1 \in S \) and let

\[
\mathcal{A} = \pi(f_1) = \{ A_1, A_2, A_3, \ldots \}.
\]

Partition each \( A_i \in \mathcal{A} \) such that \( A_i = A_i^+ \cup A_i^- \), \( |A_i^+| = |A_i^-| = \aleph_0 \). Let \( \mathcal{B} \) be the partition of \( X \)
given by

\[ B = \{ A_1', A_1'' \cup A_2', A_2'' \cup A_3', \ldots \}. \]

Since \( B \) is a partition of \( X \) into \( \aleph_0 \) sets, each of cardinality \( \aleph_0 \), \( B \in \pi(S) \), and so there exists \( f_2 \in S \) with \( \pi(f_2) = B \). Suppose \( l \in L_{f_1, f_2} \) that is \( lf_1 = lf_2 \) and let \( l(A_1) = x \). Then because of the choice of \( B \) we have the following chain of equalities:

\[ x = lf_1(A_1) = lf_1(A_1'') = lf_2(A_1'') = lf_2(A_2') = lf_1(A_2') = \ldots \]

thus

\[ x = lf_1(A_1) = lf_1(A_2) = \ldots, \]

that is \( R(lf_1) = \{ x \} \) and \( lf_1 \) is a constant in \( S \), contradicting the construction of \( S \), so that \( L_{f_1, f_2} = \emptyset \).

Recall that \( D_{f_1, f_2} \) and \( D_{f_1, f_2} \) (Notation 2.6) were defined for an arbitrary \( \mathcal{G}_X \)-normal semigroup \( S \) \((f_1, f_2 \in S)\). The following remark is an immediate consequence of the definition of \( D_{f_1, f_2} \).

**Remark 3.8.** Let \( f_1, f_2 \in S \), then \( D_{f_1, f_2} \subseteq \mathcal{D}_2 \).

We proceed with two results deriving relationships between point left ideals and function left ideals.

**Result 3.9.** Let \( f_1 \) and \( f_2 \) be distinct elements of \( S \), and \( L_{f_1, f_2} \neq \emptyset \). Then

\[ L_{f_1, f_2} = \bigcap_{A \in D_{f_1, f_2}} L_A. \]

**Proof.** Let \( l \in L_{f_1, f_2} \), that is \( lf_1 = lf_2 \) and so for each \( x \in D_{f_1, f_2} \) we have \( lf_1(x) = lf_2(x) \) (recall \( f_1(x) \neq f_2(x) \)) so \( l \in L_{f_1(x), f_2(x)} \) and since this is true for each \( x \in D_{f_1, f_2} \) we conclude

\[ l \in \bigcap_{x \in D_{f_1, f_2}} L_{f_1(x), f_2(x)} = \bigcap_{A \in D_{f_1, f_2}} L_A, \quad \text{or} \quad L_{f_1, f_2} \subseteq \bigcap_{A \in D_{f_1, f_2}} L_A. \]

Conversely, let \( l \in \bigcap_{A \in D_{f_1, f_2}} L_A \), then for each \( x \in D_{f_1, f_2} \), \( lf_1(x) = lf_2(x) \). Now for each \( y \notin D_{f_1, f_2} \) we have \( f_1(y) \neq f_2(y) \), so we deduce \( lf_1 = lf_2 \). That is, \( l \in L_{f_1, f_2} \) and \( \bigcap_{A \in D_{f_1, f_2}} L_A \subseteq L_{f_1, f_2} \), which proves the desired equality.
Result 3.10  Given an $A \in \mathcal{P}_2$, there exist $f_1$ and $f_2$ in $S$ such that

$$\mathcal{L}_A = \mathcal{L}_{f_1,f_2}.$$ 

Proof.  On account of Result 3.9 it is sufficient to construct $f_1$ and $f_2$ such that

$$D_{f_1,f_2} = \{A\}.$$ 

Choose an $f$ in $\mathcal{L}_A$ (Lemma 3.3) and let $f(A) = z$. Let $A = \{a_1, a_2\}$. Since $S$ is transitive (Result 1.3) there exists $g$ in $S$ such that $g(z) = a_1$. Let $h = (a_1, a_2)$ and

$$f_1 = g f; \quad f_2 = h f_1 h^{-1}.$$ 

Since $h$ moves only points in $A$ and $f_1 \in \mathcal{L}_A$ ($\mathcal{L}_A$ is a left ideal), we conclude $f_2 = h f_1$.

For each $x \in X \setminus f_1^{-1}(A)$ we have:

$$f_1(x) = h f_1(x) = f_2(x),$$

so $\mathcal{D}_{f_1,f_2} \subseteq f_1^{-1}(A)$. Now if $x \in f_1^{-1}(A)$, that is $f_1(x) = a_i$, $i = 1, 2$, then

$$f_1(x) = a_i \neq h(a_i) = h f_1(x) = f_2(x),$$

hence $\mathcal{D}_{f_1,f_2} \supseteq f_1^{-1}(A)$. We conclude

$$\mathcal{D}_{f_1,f_2} = f_1^{-1}(A).$$

Thus

$$D_{f_1,f_2} = \{(f_1(x), f_2(x)) : x \in \mathcal{D}_{f_1,f_2}\} \quad \text{(Notation 2.6)}$$

$$= \{(f_1(x), f_2(x)) : x \in f_1^{-1}(A)\}$$

$$= \{(a_i, h(a_i)) : i = 1, 2\}$$

$$= \{(a_1, a_2)\}$$

$$= \{A\},$$

as required. \qed

Result 3.11.  Given distinct $f_1$ and $f_2$ in $S$, $\mathcal{L}_{f_1,f_2}$ is a maximal function left ideal if and only if $|D_{f_1,f_2}| = 1$.

Proof.  Let $\mathcal{L}_{f_1,f_2}$ be a maximal function left ideal and suppose $A, B \in D_{f_1,f_2}, A \neq B$. 
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Then $A, B \in \mathcal{P}_2$ (Remark 3.8). Hence

$$\mathcal{L}_{f_1, f_2} = \bigcap_{C \in D_{f_1, f_2}} \mathcal{L}_C \quad \text{(Result 3.9)}$$

$$\subseteq \mathcal{L}_A \cap \mathcal{L}_B$$

$$\subseteq \mathcal{L}_A \quad \text{(Lemma 3.4)}$$

$$= \mathcal{L}_{g_1, g_2} \quad \text{(Result 3.10),}$$

for some distinct $g_1, g_2 \in S$, contradicting the maximality of $\mathcal{L}_{f_1, f_2}$. Hence $|D_{f_1, f_2}| = 1$.

Conversely, suppose $D_{f_1, f_2} = \{A\}$, some $A \in \mathcal{P}_2$, while there exists a function left ideal $\mathcal{L}_{g_1, g_2}$ ($g_1, g_2 \in S$) such that

$$\mathcal{L}_{g_1, g_2} \supseteq \mathcal{L}_{f_1, f_2}.$$ 

Since $\mathcal{L}_{g_1, g_2} = \bigcap_{B \in D_{g_1, g_2}} \mathcal{L}_B$ (Result 3.9) we have

$$\bigcap_{B \in D_{g_1, g_2}} \mathcal{L}_B = \mathcal{L}_{g_1, g_2} \supseteq \mathcal{L}_{f_1, f_2} = \mathcal{L}_A \quad \text{(Result 3.9 again)},$$

and so Lemma 3.4 ensures $D_{g_1, g_2} = \{A\}$, that is

$$\mathcal{L}_{g_1, g_2} = \mathcal{L}_A = \mathcal{L}_{f_1, f_2}. \quad \square$$

Corollary 3.12. Given $f_1$ and $f_2$ is $S$, $\mathcal{L}_{f_1, f_2}$ is a maximal left function ideal if and only if $\mathcal{L}_{f_1, f_2} = \mathcal{L}_A$, some $A \in \mathcal{P}_2$.

Proof. Follows from Results 3.9 and 3.11. \square

We show now that each automorphism $\phi$ of $S$ permutes point left ideals.

Result 3.13. Given $A \in \mathcal{P}_2$,

$$\phi(\mathcal{L}_A) = \mathcal{L}_B,$$

for some $B \in \mathcal{P}_2$. 

\square
\textbf{Proof.} Choose \( f_1 \) and \( f_2 \) in \( S \) such that \( L_{f_1, f_2} = L_A \) (Result 3.10), then

\[
\phi(L_A) = \phi(L_{f_1, f_2}) = \phi(\{l : lf_1 = lf_2\})\\
= \{\phi(l) : \phi(lf_1) = \phi(lf_2)\}\\
= \{\phi(l) : \phi(l)\phi(f_1) = \phi(l)\phi(f_2)\}\\
= \{l : \phi(f_1) = \phi(f_2)\}\\
= L_{\phi(f_1), \phi(f_2)}.
\]

Now Corollary 3.12 ensures \( L_{f_1, f_2} \) is a maximal function left ideal, hence \( L_{\phi(f_1), \phi(f_2)} \) (\( = \phi(L_{f_1, f_2}) \)) is a maximal function left ideal, so there exists \( B \in \mathcal{P}_2 \) such that

\[
L_{\phi(f_1), \phi(f_2)} = L_B \quad \text{(Corollary 3.12)}.
\]

We conclude

\[
\phi(L_A) = L_{\phi(f_1), \phi(f_2)} = L_B. \quad \square
\]

Define a map \( \mu : \{L_A : A \in \mathcal{P}_2\} \to \{L_A : A \in \mathcal{P}_2\} \)

via \( \mu(L_A) = \phi(L_A) \), each \( L_A \subseteq S \).

\textbf{Lemma 3.14.} \( \mu \) is a bijection.

\textbf{Proof.} That \( \mu \) is a mapping is the content of Result 3.13. Similarly by considering the automorphism \( \phi^{-1} \) we define a map

\[
\xi : \{L_A : A \in \mathcal{P}_2\} \to \{L_A : A \in \mathcal{P}_2\}
\]

via \( \xi(L_A) = \phi^{-1}(L_A) \), each \( L_A \subseteq S \). Certainly, \( \xi \) is the inverse of \( \mu \) and so \( \mu \) is a bijection. \quad \square

We now define a map \( \lambda : \mathcal{P}_2 \to \mathcal{P}_2 \) via \( \lambda(A) = B \), where \( \mu(L_A) = L_B \), each \( A \in \mathcal{P}_2 \).

It is clear that

\[
\lambda = \delta^{-1} \mu \delta,
\]
and so Lemmas 3.5 and 3.14 ensure \( \lambda \) is a bijection of \( S_2 \). We call \( \lambda \) the bijection of \( S_2 \) associated with \( \phi \).

We show that \( \lambda \) is induced by a bijection \( h \) of \( X \), that is
\[
\lambda(A) = h(A),
\]
for each \( A \in S_2 \). Note here that not every bijection of \( S_2 \) is induced, as shown in Example 3.15 below.

**Example 3.15.** Fix \( A \) and \( C \) in \( S_2 \), \( A \neq C \) and let \( \lambda \) be a bijection of \( S_2 \), which interchanges \( A \) and \( C \) and the identity otherwise. Choose \( B \in S_2 \), \( B = \{x, y\} \) such that \( x \in A \cap C \) and \( y \in X \setminus (A \cup C) \). Note \( A \cap B = \{x\} \) and \( B \cap C = \emptyset \). Suppose \( \lambda \) is induced by \( h \in S_X \), then
\[
h(x) = h(A \cap B) = h(A) \cap h(B) = \lambda(A) \cap \lambda(B) = C \cap B = \emptyset.
\]
Thus \( \lambda \) is not induced. \( \square \)

Observe that in the example above we had \( \lambda \), a bijection of \( S_2 \), such that
\[
|A \cap B| \neq |\lambda(A) \cap \lambda(B)|,
\]
for some \( A, B \) in \( S_2 \). This leads us to a criterion for a bijection \( \lambda \) of \( S_2 \) to be induced.

**Result 3.16.** Let \( \lambda \) be a bijection of \( S_2 \). Then \( \lambda \) is induced if and only if
\[
|A \cap B| = |\lambda(A) \cap \lambda(B)|,
\]
for every \( A, B \in S_2 \).

**Proof.** If \( \lambda \) is induced by an \( h \in S_X \), then for every \( A, B \in S_2 \),
\[
|A \cap B| = |h(A \cap B)| = |h(A) \cap h(B)| = |\lambda(A) \cap \lambda(B)|.
\]
For the converse suppose that \( \lambda \) is a bijection of \( S_2 \) such that for every \( A, B \in S_2 \)
\[
|A \cap B| = |\lambda(A) \cap \lambda(B)|.
\]
(\( \ast \))

We show that \( \lambda \) is induced. This is done in the following three steps.

1. Given \( x \in X \) there exists a unique \( y \in X \) such that for every \( A, B \in S_2 \) with \( A \cap B = \{x\} \) we have \( \lambda(A) \cap \lambda(B) = \{y\} \).

Take a pair \( A, B \) in \( S_2 \) with \( A \cap B = \{x\} \), then by the assumption (\( \ast \)) \( \lambda(A) \cap \lambda(B) = \{y\} \), for some \( y \in X \).

Take any other pair \( C, D \) in \( S_2 \) with \( |C \cap D| = 1 \) and let \( \mathcal{F} \subseteq S_2 \) be such that:

(a) for every distinct \( F_1, F_2 \in \mathcal{F} \), \( |F_1 \cap F_2| = 1 \);

(b) for any \( F \in \mathcal{F} \), \( |A \cap F| = |B \cap F| = |C \cap F| = |D \cap F| = 1 \).

We show:
\[
C \cap D = \{x\} \text{ iff there exists an } \mathcal{F} \text{ (as described above) with } |\mathcal{F}| = |X|.
\]

Let \( A \cup B \cup C \cup D = E \), then \( |E| \leq 8 \) and \( |X \setminus E| = |X| \).

Assume firstly that \( C \cap D = \{x\} \) and let \( \mathcal{F} = \{\{x, y\}: y \in X \setminus E\} \). Then \( \mathcal{F} \) satisfies (a) and (b) and \( |\mathcal{F}| = |X \setminus E| = |X| \).
For the converse assume \( C \cap D = \{ z \} \), \( z \neq x \) and \( \mathcal{F} \subseteq \mathcal{P}_2 \) satisfies (\( \alpha \)) and (\( \beta \)). For each \( F \in \mathcal{F} \) we have \(|E \cap F| > 1\). (If not, then

\[
|E \cap F| = |(A \cup B \cup C \cup D) \cap F|
\]

\[
= |(A \cap F) \cup (B \cap F) \cup (C \cap F) \cup (D \cap F)| \leq 1.
\]

Using condition (\( \beta \)) we conclude:

\[
A \cap F = B \cap F = C \cap F = D \cap F = A \cap B = \{ x \},
\]

or \( C \cap D = \{ x \} \), a contradiction).

Define a map \( \nu: \mathcal{F} \to \mathcal{P}(E) \), where \( \mathcal{P}(E) \) is the power set of \( E \), via \( \nu(F) = E \cap F \), each \( F \in \mathcal{F} \). We show \( \nu \) is 1–1. Suppose \( F_1, F_2 \in \mathcal{F} \) with \( \nu(F_1) = \nu(F_2) \). Then

\[
1 < |E \cap F_1| = |E \cap F_1 \cap F_2| \leq |F_1 \cap F_2|,
\]

so that \(|F_1 \cap F_2| > 1\), thus \( F_1 = F_2 \) (condition (\( \alpha \))). However \( \mathcal{P}(E) \) is finite, so \(|\mathcal{F}| \leq |\mathcal{P}(E)| < \aleph_0 \), or \(|\mathcal{F}| < |X|\). We conclude \( C \cap D = \{ x \} \).

Observe now that the definition of the set \( \mathcal{F} \) depends on the sets \( A, B, C \) and \( D \). We denote this dependence by \( \mathcal{F} = \mathcal{F}(A, B, C, D) \). Hence \( C \cap D = \{ x \} \) iff \( \exists \mathcal{F}(A, B, C, D) \) with \(|\mathcal{F}(A, B, C, D)| = |X| \) iff \( \exists \mathcal{F}(\lambda(A), \lambda(B), \lambda(C), \lambda(D)) \) with \(|\mathcal{F}(\lambda(A), \lambda(B), \lambda(C), \lambda(D))| = |X| \) (assumption (\( \ast \)))

\[
\text{iff } \lambda(C) \cap \lambda(D) = \{ y \}.
\]

Now define a map

\[
h: X \to X \text{ via } \{h(x)\} = \lambda(A) \cap \lambda(B), \text{ where } \{x\} = A \cap B, \text{ for } A, B \in \mathcal{P}_2 \text{ and each } x \in X.
\]

2. \( h \) is a bijection of \( X \).

That \( h \) is well-defined is the content of step 1. Observe that the bijection \( \lambda^{-1} \) of \( \mathcal{P}_2 \) is associated with the automorphism \( \phi^{-1} \). By considering \( \phi^{-1} \) and \( \lambda^{-1} \) instead of \( \phi \) and \( \lambda \) we define a map \( k: X \to X \) via \( \{k(x)\} = \lambda^{-1}(A) \cap \lambda^{-1}(B) \), where \( \{x\} = A \cap B \), for \( A, B \in \mathcal{P}_2 \) and each \( x \in X \). Then for each \( x \in X \)

\[
\{kh(x)\} = k(\lambda(A) \cap \lambda(B)), \text{ where } A \cap B = \{x\}
\]

\[
= \lambda^{-1}(A) \cap \lambda^{-1}(B)
\]

\[
= A \cap B
\]

\[
= \{x\}.
\]

Similarly we can show \( hk(x) = x \), for each \( x \in X \). Thus \( k \) is the inverse of \( h \), and so \( h \) is a bijection of \( X \).
3. \( \lambda \) is induced by \( h \).

To show \( \lambda \) is induced by \( h \) we must show \( \lambda(A) = h(A) \) for each \( A \in \mathcal{P}_2 \). From the definition of \( h \) we at once have \( h(A) \subseteq \lambda(A) \). Take \( y \in \lambda(A) \) and let \( B \in \mathcal{P}_2 \) be such that \( \lambda(A) \cap \lambda(B) = \{ y \} \). Then \( A \cap B = \{ x \} \), some \( x \in A \), so \( h(x) = y \) and \( h(A) \supseteq \lambda(A) \). The equality follows. \( \square \)

**Remark 3.17.** In view of Result 3.16 our aim now is to show that for every \( A, B \in \mathcal{P}_2 \)
\[
|A \cap B| = |\lambda(A) \cap \lambda(B)|
\]

(\*) where \( \lambda \) is the bijection of \( \mathcal{P}_2 \) associated with \( \phi \) as defined prior to Example 3.15.
Observe that (\*) is equivalent to the statement
\[
|A \cap B| = 1 \quad \text{if and only if} \quad |\lambda(A) \cap \lambda(B)| = 1,
\]

(\**\)** for each \( A, B \in \mathcal{P}_2 \).

Indeed (\*) certainly implies (\**\**). We show the reverse implication.

Assume (\**\**) holds. If \( |A \cap B| = 2 \), that is \( A = B \), then \( \lambda(A) = \lambda(B) \), and so \( |\lambda(A) \cap \lambda(B)| = 2 \).
If \( |A \cap B| = 1 \), then by our assumption \( |\lambda(A) \cap \lambda(B)| = 1 \). The case \( |A \cap B| = 0 \) follows by elimination. \( \square \)

The next lemma illustrates the fact that the existence of a transformation \( f \) in \( S \) which is not 1–1 provides an extensive variety of elements in \( \pi(S) \).

**Lemma 3.18.** Given \( B_1, B_2 \subseteq X \) with \( B_1 \cap B_2 = \emptyset \) and \( |B_1| = |B_2| = 3 \) there exists an \( \mathcal{A} \in \pi(S) \) with \( B_1 \subseteq A_1 \in \mathcal{A} \), \( B_2 \subseteq A_2 \in \mathcal{A} \).

**Proof.** Suppose that there exists a transformation \( f \) in \( S \) such that:
\[ C_1, C_2 \in \pi(f) \quad \text{and} \quad |C_1|, |C_2| \geq 3. \]
Choose a bijection \( p \) of \( X \) with
\[ B_1 \subseteq p(C_1) \quad \text{and} \quad B_2 \subseteq p(C_2). \]
Certainly \( pf p^{-1} \in S \). Let
\[ \mathcal{A} = \pi(pf p^{-1})(= p(\pi(f))), A_1 = p(C_1) \quad \text{and} \quad A_2 = p(C_2), \]
then \( A_1, A_2 \in \mathcal{A} \in \pi(S) \) and \( B_1 \subseteq A_1, B_2 \subseteq A_2 \).

To construct such an \( f \) as used above we first show that there exists a \( g \) in \( S \) such that
\[ g(x_1) = g(x_2) = g(x_3) = x_1, \quad \text{for some distinct} \ x_1, x_2, x_3 \in X. \]
Choose a $t$ in $S$ not 1–1 and let $x, x_1, x_2 \in X$ be such that $$t(x_1) = t(x_2) = x.$$ We assume $x = x_1$ (for if $x \neq x_1$ choose $s \in S$ such that $s(x) = x_1$ (Result 1.3) and replace $t$ by $st$). Let $x_4 \in R(t) \setminus \{t^{-1}(x_1)\}$ (note: $R(t) \setminus \{t^{-1}(x_1)\} \neq \emptyset$, else $t^2$ is a constant in $S$) and let $x_3 \in X$ such that $t(x_3) = x_4$. Then $$g = (x_2, x_4)t(x_2, x_4)t$$ is such that $g(x_1) = g(x_2) = g(x_3) = x_1$. 

To accomplish the construction of the above $f$ choose distinct $z_1, z_2, z_3$ in $R(g) \setminus \{g^{-1}(x_1)\}$ together with $y_1, y_2, y_3 \in X$ such that $g(y_i) = z_i$, $i = 1, 2, 3$. Let $$k = (x_1, z_1)(x_2, z_2)(x_3, z_3) \in \mathcal{G}_X \quad \text{and} \quad f = kgk^{-1}g.$$ Let $kg(z_1) = z_4$. Then $$f(x_1) = f(x_2) = f(x_3) = z_4$$ and $$f(y_1) = f(y_2) = f(y_3) = z_1.$$ Now $z_1 \neq z_4$ (else $kg(z_1) = z_4$ implies $g(z_1) = z_1$ or $z_1 \in g^{-1}(x_1)$, contrary to the choice of $z_1$). Let $C_1 = f^{-1}(z_1), C_2 = f^{-1}(z_4)$. Then $|C_1|, |C_2| \geq 3$ and $C_1, C_2 \in \pi(f)$ as required. \hfill $\square$

**Remark 3.19.** It easily follows from Lemma 3.18 that

$$\mathcal{L}_A \cap \mathcal{L}_B \neq \emptyset,$$

for every $A, B \in \mathcal{P}_2$. \hfill $\square$

**Lemma 3.20.** Let $A, B \in \mathcal{P}_2$, $A \neq B$. Then $|A \cap B| = 1$ iff there is a $C$ in $\mathcal{P}_2$, $C \neq A$ or $B$, such that $\mathcal{L}_A \cap \mathcal{L}_B \subseteq \mathcal{L}_C$.

**Proof.** Assume $|A \cap B| = 1$ and let $C = (A \cup B) \setminus (A \cap B)$. For each $l$ in $\mathcal{L}_A \cap \mathcal{L}_B$ (Remark 3.19):

$$l(A) = l(A \cap B) = l(B) = l(A \cup B) = l(C),$$

so that $l \in \mathcal{L}_C$ and $\mathcal{L}_A \cap \mathcal{L}_B \subseteq \mathcal{L}_C$.

For the converse suppose $A \cap B = \emptyset$ and $C \in \mathcal{P}_2$ is distinct from $A$ and $B$. Let $C = \{c_1, c_2\}$. Since $|A \cap C| \leq 1$ and $|B \cap C| \leq 1$ assume without loss of generality that $c_1 \in X \setminus B$ and $c_2 \in X \setminus A$. Choose

$$\mathcal{A} \in \pi(S) \quad \text{with} \quad A \cup \{c_1\} \subseteq A_1 \in \mathcal{A}, \ B \cup \{c_2\} \subseteq A_2 \in \mathcal{A} \quad \text{and} \quad A_1 \neq A_2 \quad \text{(Lemma 3.18)}.$$
If \( l \in S \) has \( \pi(l) = \mathcal{A} \), then \( l \in (\mathcal{L}_A \cap \mathcal{L}_B) \setminus \mathcal{L}_C \).

This confirms that \( |A \cap B| = 1 \).

**Lemma 3.21.** Let \( A, B \) and \( C \) be distinct elements of \( \mathcal{P}_2 \). Then

\[
\mathcal{L}_A \cap \mathcal{L}_B \subseteq \mathcal{L}_C \iff \mathcal{L}_{\lambda(A)} \cap \mathcal{L}_{\lambda(B)} \subseteq \mathcal{L}_{\lambda(C)}.
\]

**Proof.** Observe that \( \mathcal{L}_A \cap \mathcal{L}_B \neq \emptyset \) (Remark 3.19) and

\[
\mathcal{L}_A \cap \mathcal{L}_B \subseteq \mathcal{L}_C \iff \phi(\mathcal{L}_A \cap \mathcal{L}_B) \subseteq \phi(\mathcal{L}_C).
\]

Now

\[
\phi(\mathcal{L}_A \cap \mathcal{L}_B) = \phi(\mathcal{L}_A) \cap \phi(\mathcal{L}_B) = \mathcal{L}_{\lambda(A)} \cap \mathcal{L}_{\lambda(B)},
\]

by the definition of \( \lambda \). Also \( \phi(\mathcal{L}_C) = \mathcal{L}_{\lambda(C)} \), so that

\[
\phi(\mathcal{L}_A \cap \mathcal{L}_B) \subseteq \phi(\mathcal{L}_C) \iff \mathcal{L}_{\lambda(A)} \cap \mathcal{L}_{\lambda(B)} \subseteq \mathcal{L}_{\lambda(C)},
\]

and the desired equivalence is established. \( \square \)

**Result 3.22.** Given \( A \) and \( B \) in \( \mathcal{P}_2 \),

\[
|A \cap B| = 1 \text{ if and only if } |\lambda(A) \cap \lambda(B)| = 1.
\]

**Proof.** We have:

\[
|A \cap B| = 1 \iff \exists C \neq A \text{ or } B \text{ such that } \mathcal{L}_A \cap \mathcal{L}_B \subseteq \mathcal{L}_C \quad \text{(Lemma 3.20)}
\]

\[
\iff \exists \lambda(C) \neq \lambda(A) \text{ or } \lambda(B) \text{ such that } \mathcal{L}_{\lambda(A)} \cap \mathcal{L}_{\lambda(B)} \subseteq \mathcal{L}_{\lambda(C)}
\]

\[
(\lambda \text{ is a bijection and Lemma 3.21})
\]

\[
\iff |\lambda(A) \cap \lambda(B)| = 1 \quad \text{(Lemma 3.20 again)}.
\]

From Results 3.16, 3.22 and Remark 3.17 we readily deduce

**Result 3.23.** \( \lambda \) is induced by a bijection of \( X \). \( \square \)

Now we are ready to show that a constant-free \( G_X \)-normal semigroup containing a transformation which is not 1–1 (that is a semigroup of Type 3), possesses only inner automorphisms.

**Result 3.24.** Let \( S \) be a constant-free \( G_X \)-normal semigroup containing a transformation which is not 1–1. Then each automorphism \( \phi \) of \( S \) is inner, that is for some \( h \in G_X \)

\[
\phi(f) = hfh^{-1}, \text{ for each } f \in S.
\]

**Proof.** Let \( h \) be the bijection which induces \( \lambda \) (Result 3.23). In what follows we use
the fact that for any distinct \( x_1, x_2 \in X \)

\[ \phi(L_{(x_1, x_2)}) = L_{h(x_1), h(x_2)}. \]

Take an arbitrary \( f \in S, x \in X \) and let \( y \in X \) with \( f(x) \neq f(y) \) (that is \( f \notin L_{(x, y)} \)). Then

\[ \phi(L_{(f(x), f(y))}) = L_{(hf(x), hf(y))}. \]

Let \( \phi(g) \in \phi(L_{(f(x), f(y))}) \). Then \( g \in L_{(f(x), f(y))} \) or \( g f(x) = g f(y) \). It follows that \( \phi(g f) \in L_{(h(x), h(y))} \), hence

\[ \phi(g) \phi(f) h(x) = \phi(g) \phi(f) h(y). \]

Note that \( f \notin L_{(x, y)} \) implies \( \phi(f) \notin \phi(L_{(x, y)}) \) or \( \phi(f) \notin L_{(h(x), h(y))} \), that is

\[ \phi(f) h(x) \neq \phi(f) h(y). \]

Thus \( \phi(g) \in L_{(\phi(f) h(x), \phi(f) h(y))} \) and we conclude

\[ \phi(L_{(f(x), f(y))}) \subseteq L_{(\phi(f) h(x), \phi(f) h(y))}. \]

This in turn implies

\[ L_{(hf(x), hf(y))} \subseteq L_{(\phi(f) h(x), \phi(f) h(y))}. \]

Hence \( \{ hf(x), hf(y) \} = \{ \phi(f) h(x), \phi(f) h(y) \} \) (Lemma 3.4).

Since the choice of \( y \) is independent of \( x \) (providing \( y \neq x \)) we conclude

\[ \phi(f) h(x) = hf(x), \quad \text{for each} \quad x \in X, \]

so that

\[ \phi(f) = hf h^{-1}, \quad \text{as required.} \]

\[ \square \]

**Conclusion**

We return to

**Theorem 1.1.** Every automorphism of a \( G_X \)-normal semigroup \( S \) is inner.

**Proof.** If \( S \) is a semigroup of Type 1, that is, contains a constant transformation, then we appeal to Sullivan [12, Theorem 1].

If \( S \) is a semigroup of Type 2, that is, a semigroup of 1–1 transformations, the result is given in 2.14 and 1.4.

If \( S \) is a semigroup of Type 3, that is, a semigroup containing a transformation which is not 1–1, then the result is given in 3.24.

This completes the proof of Theorem 1.1. \[ \square \]
Remark. If $X$ is a finite set and $S$ is a semigroup of transformations of $X$ which is not contained in $\mathcal{G}_X$, then $S$ is $\mathcal{G}_X$-normal if and only if all automorphisms of $S$ are inner [13].

However, this is not the case for an infinite set $X$. While, as we showed, every $\mathcal{G}_X$-normal semigroup $S$ has only inner automorphisms, there are examples [5] of semigroups which are neither subsemigroups of $\mathcal{G}_X$, nor $\mathcal{G}_X$-normal, yet have only inner automorphisms.

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