

AUTOMORPHISMS AND RANGE FAMILIES  
OF TRANSFORMATION SEMIGROUPS

INESSA LEVI

The problem of describing all automorphisms of a given semigroup of transformations of a set  $X$  has interested a number of mathematicians in the past fifty years. In 1937 Schreier [10] showed that every automorphism of the full transformation semigroup  $T_X$  is *inner* (that is, acts as a conjugation by some bijection of  $X$ ). In 1952 Mal'cev [7] generalized this result by showing that every ideal of  $T_X$  has only inner automorphisms. More recently Symons [11] showed that all automorphisms of any  $G_X$ -normal semigroup (that is, invariant under a conjugation by any bijection of  $X$ ) over a *finite* set  $X$  are inner, while Schein [9] produced the same result for  $G_X$ -normal semigroups of one-to-one transformations over an *infinite* set  $X$ . (See [2] for the special case of Baer-Levi semigroups.)

Chapters 2 and 3 of this thesis constitute a contribution towards the solution of the problem of describing all automorphisms of a given semigroup of transformations of an *infinite* set  $X$ . In Chapter 2 (see also [4]) we extend the well-known result from group theory, namely that any normal group of bijections of an infinite set  $X$  has only inner automorphisms, to an analogous one in semigroup theory. We show that any  $G_X$ -normal semigroup of transformations of an infinite set  $X$  has only inner automorphisms. Our purpose in Chapter 3 (see also [3]) is to offer a complete description of all automorphisms of an arbitrary *Croisot-Teissier*

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semigroup [1], a task suggested by Schein. In this joint work with O'Meara and Wood a rich variety of automorphisms is found, ranging from inner, to "locally" inner, to thoroughly outer. We also present a description of Green's relations on Croisot-Teissier semigroups.

In Chapter 4 (and in [5]) we are concerned with the problem of a characterisation of all subsets of the power set of  $X$ ,  $\mathcal{P}_X$ , which serve as sets of ranges of semigroups of transformations of  $X$ . This problem was suggested by Schein and to our knowledge has been solved only for the case of monogenic semigroups of partial transformations by Olonichev [8]. We define a *normal* subset of  $\mathcal{P}_X$  and characterise all normal subsets of  $\mathcal{P}_X$  which serve as sets of ranges of semigroups of total transformations of  $X$ . In particular, we give necessary and sufficient conditions for a subset of  $\mathcal{P}_X$  to be the set of ranges of a  $G_X$ -normal and a constant-free  $G_X$ -normal semigroup of total transformations.

In Chapter 5 (and in [6]) for a particular normal subset of  $\mathcal{P}_X$  we give necessary and sufficient conditions for an order-automorphism to be determined by a bijection of  $X$  (that is, *induced*). We then characterise those normal subsets of  $\mathcal{P}_X$  for which all order-automorphisms are induced. Apart from being of independent interest, this problem is connected with the study of automorphisms of transformation semigroups. For if an automorphism  $\phi$  of a transformation semigroup  $S$  is inner, then  $\phi$  produces an induced order-automorphism of the set  $R(S)$  of ranges of all transformations in  $S$ . On the other hand, in instances where an automorphism  $\phi$  of  $S$  yields an order-automorphism of  $R(S)$ , the knowledge that all order-automorphisms of  $R(S)$  are induced can be a first step in showing that  $\phi$  is inner.

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Department of Mathematics,  
University of Canterbury,  
Christchurch 1,  
New Zealand.