numbers defined by $a_n = \frac{1}{2}(a_{n-1}^2 + 1)$ for $n > 1$, with $a_1 = 3$. Show that

$$\left[\left(\sum_{k=1}^{n} \frac{a_k}{1+a_k}\right)\left(\sum_{k=1}^{n} \frac{1}{a_k(1+a_k)}\right)\right]^{1/2} \le \frac{1}{4}\left(\frac{a_1 + a_n}{\sqrt{a_1 a_n}}\right).$$

11443. *Proposed by Eugen Ionascu, Columbus State University, Columbus, GA.* Consider a triangle ABC with circumcenter O and circumradius R. Denote the distances from O to the sides AB, BC, CA, respectively, by x, y, z. Show that if ABC is acute then $R^3 - (x^2 + y^2 + z^2)R = 2xyz$, and $(x^2 + y^2 + z^2)R - R^3 = 2xyz$ otherwise.

11444. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu", Bîrlad, Romania.* Let k and s be positive integers with $s \le k$. Let $f(n) = n - s\lfloor n/k \rfloor$. For $j \ge 0$, let f^j denote the j-fold composition of f, taking f^0 to be the identity function. Show that

$$\sum_{j=0}^{\infty} \left\lfloor \frac{f^j(n)}{k} \right\rfloor = -\left\lfloor \frac{q-n}{s} \right\rfloor,$$

where $q = \min\{k-1, n\}$.

11445. *Proposed by H. A. ShahAli, Tehran, Iran.* Given $a, b, c > 0$ with $b^2 > 4ac$, let $\langle \lambda_n \rangle$ be a sequence of real numbers, with $\lambda_0 > 0$ and $c\lambda_1 > b\lambda_0$. Let $u_0 = c\lambda_0$, $u_1 = c\lambda_1 - b\lambda_0$, and for $n \ge 2$ let $u_n = a\lambda_{n-2} - b\lambda_{n-1} + c\lambda_n$. Show that if $u_n > 0$ for all $n \ge 0$, then $\lambda_n > 0$ for all $n \ge 0$.

SOLUTIONS

Sums and Powers, Set Counting, and Coefficient Tracking

11274 [2007, 165]. *Proposed by Donald Knuth, Stanford University, Stanford, CA.* Prove that for nonnegative integers m and n,

$$\sum_{k=0}^{\infty} 2^k \binom{2m-k}{m+n} = 4^m - \sum_{j=1}^{n} \binom{2m+1}{m+j}.$$

Solution I by Julian Hook, Indiana University, Bloomington, IN. We show that both sides equal $\sum_{r=m+n+1}^{2m+1} \binom{2m+1}{r}$, the number of subsets of the set $\{1, \dots, 2m+1\}$ having size at least $m + n + 1$. Such a subset may be constructed by picking an element z to be the $(m + n + 1)$th-smallest element chosen, picking $m + n$ elements below z, and picking any subset of the elements above z. With $k = 2m + 1 - z$, the value of k indexes the choices for z. Since there are $2m - k$ elements below z from which to pick $m + n$, and k elements above z from which to pick arbitrarily, the left side of the equation counts the specified subsets.

To count another way, note first that exactly half of the $2 \cdot 4^m$ subsets of $\{1, \dots,$ $2m + 1\}$ have size at least $m + 1$. From these, we eliminate those with sizes from $m + 1$ to $m + n$ by subtracting the sum on the right side.