On the inner automorphisms of finite transformation semigroups.

Inessa Levi

Follow this and additional works at: http://csuepress.columbusstate.edu/bibliography_faculty

Part of the Mathematics Commons

Recommended Citation

This Article is brought to you for free and open access by CSU ePress. It has been accepted for inclusion in Faculty Bibliography by an authorized administrator of CSU ePress.
ON THE INNER AUTOMORPHISMS OF FINITE TRANSFORMATION SEMIGROUPS

by INESSA LEVI

(Received 20th December 1993)

If the group of inner automorphisms of a semigroup $S$ of transformations of a finite $n$-element set contains an isomorphic copy of the alternating group $\text{Alt}_n$, then $S$ is an $S_n$-normal semigroup and all the automorphisms of $S$ are inner.


1. Introduction

Given a semigroup $S$ of transformations of a set $X_n = \{1, 2, \ldots, n\}$, denote by $G_S$ the subgroup of the symmetric group $S_n$ of all the permutations $h$ of $X_n$ satisfying $hSh^{-1} \subseteq S$. Therefore for each $h \in G_S$, the mapping $\phi_h: S \rightarrow S$ defined by $\phi_h(\alpha) = h\alpha h^{-1}$, for $\alpha \in S$, is an automorphism of $S$. Such an automorphism of $S$ is termed inner [5] and the set of all inner automorphisms of $S$, $\text{Inn} S = \{\phi_h; h \in G_S\}$, forms a subgroup of the group $\text{Aut} S$ of all automorphisms of $S$.

Observe that if $S = T_n$, the semigroup of all total transformations of $X$, then $G_S = S_n$. A subsemigroup $S$ of $T_n$ is said to be $S_n$-normal if $G_S = S_n$. In this case all the automorphisms of $S$ are inner, and $\text{Aut} S = \text{Inn} S \cong S_n$ [6].

The main result of this paper asserts that if $G_S$ contains the alternating group $\text{Alt}_n$ then $G_S = S_n$, so that $S$ is an $S_n$-normal semigroup, and $\text{Aut} S = \text{Inn} S \cong S_n$. Therefore, there is no $S \subseteq T_n$ such that $G_S = \text{Alt}_n$.

We generally use letters $h, p, g$ to denote permutations of $X_n$, and $\alpha, \beta, \gamma, \delta$ to denote non-permutations in $T_n$. In the following series of results we prove the theorem stated below.

Theorem. Let $S$ be a subsemigroup of $T_n$, $n \geq 3$. If the group $\text{Inn} S$ contains a subgroup $G$ isomorphic to $\text{Alt}_n$, then $\text{Aut} S = \text{Inn} S \cong S_n$, and $S$ is an $S_n$-normal semigroup.

Given $\alpha \in T_n$ and a subgroup $G$ of $S_n$, let $\langle \alpha; G \rangle = \langle \{h\alpha h^{-1}; h \in G\} \rangle$ be the subsemigroup of $T_n$ generated by all the conjugates of $\alpha$ by the elements of $G$. Observe that if $\beta \in \langle \alpha; G \rangle$, then $\beta = h_1 \alpha h_1^{-1} h_2 \alpha h_2^{-1} \ldots h_k \alpha h_k^{-1}$ for some $h_1, h_2, \ldots, h_k \in G$, and so for any $h \in G$, $h\beta h^{-1} = hh_1 \alpha h_1^{-1} h^{-1} hh_2 \alpha h_2^{-1} h^{-1} \ldots hh_k \alpha h_k^{-1} = (hh_1) \alpha (hh_1)^{-1} (hh_2) \alpha (hh_2)^{-1} \ldots (hh_k) \alpha (hh_k)^{-1} \in \langle \alpha; G \rangle$. Therefore $\langle \beta; G \rangle \subseteq \langle \alpha; G \rangle$. 

27
Lemma 1. Let $G_1 \leq G_2 \leq S_n$ and $[G_2 : G_1] = 2$. Let $\alpha \in T_n - S_n$. Then $\langle \alpha : G_1 \rangle = \langle \alpha : G_2 \rangle$ if and only if there is an $h \in G_2 - G_1$ such that $h \alpha h^{-1} \in \langle \alpha : G_1 \rangle$.

Proof. If $\langle \alpha : G_1 \rangle = \langle \alpha : G_2 \rangle$ then for any $h \in G_2$, $h \alpha h^{-1} \in \langle \alpha : G_1 \rangle$. To show the converse assume that $h \in G_2 - G_1$ is such that $h \alpha h^{-1} \in \langle \alpha : G_1 \rangle$. Let $p \in G_2 - G_1$. It suffices to show that $p x p^{-1} \in \langle \alpha : G_1 \rangle$. Since $h, p \in G_2 - G_1$ and $[G_2 : G_1] = 2$, we have $G_1 h = G_1 p$, so there exists $q \in G_1$, with $q = ph^{-1}$. Therefore $p x p^{-1} = q \alpha (q h^{-1})^{-1} = q h a h^{-1} q^{-1} = q \beta q^{-1} \in \langle \beta : G_1 \rangle \subseteq \langle \alpha : G_1 \rangle$, as required. \[ \square \]

The following is used to show that if $\alpha \in T_n - S_n$ then $\langle \alpha : \text{Alt}_n \rangle = \langle \alpha : S_n \rangle$.

Corollary 2. $\langle \alpha : \text{Alt}_n \rangle = \langle \alpha : S_n \rangle$ if and only if there exists an odd permutation $h$ of $X_n$ such that $h \alpha h^{-1} \in \langle \alpha : \text{Alt}_n \rangle$.

Recall that a subgroup $G$ of $S_n$ is said to be $k$-transitive if for any two $k$-subsets $A$ and $B$ of $X_n$ and any bijection $t$ from $A$ onto $B$, there exists $h \in G$ such that $h(a) = t(a)$ for every $a \in A$. We say that a subgroup $G$ of $S_n$ is $k$-block-transitive if for any two $k$-subsets $A$ and $B$ of $X_n$ there exists $h \in G$ such that $h(A) = B$. Thus any $k$-transitive semigroup is at least $k$-block-transitive. For example, Alt$_n$ is $(n - 2)$-transitive \[ [4, 10.4.6] \], and for all $1 \leq k \leq n - 1$, Alt$_n$ is $k$-block transitive.

Given a transformation $\alpha$ of $X_n$ denote by $\pi(\alpha)$ the partition of $X_n$ determined by $\alpha$ such that $a$ and $b$ are in the same class of $\pi(\alpha)$ if and only if $\langle \alpha(a) \rangle = \langle \alpha(b) \rangle$. Let $\text{im} \alpha = \alpha(X_n)$ be the image of $\alpha$. Note that if $h \in S_n$ then $\pi(h \alpha h^{-1}) = h(\pi(\alpha)) = \{h(A) : A \in \pi(\alpha)\}$, and $\text{im}(h \alpha h^{-1}) = h(\text{im} \alpha)$.

Lemma 3. Let $G \leq S_n$ be a $k$-block transitive group. Then for any $\alpha \in T_n - S_n$ with $|\text{im} \alpha| = k$, $\langle \alpha : G \rangle$ contains an idempotent $\beta$ with $\pi(\beta) = \pi(\alpha)$.

Proof. Let $\alpha_1 (= \alpha), \alpha_2, \alpha_3, \ldots \beta$ be conjugates of $\alpha$ by elements of $G$ such that $\text{im} \alpha_i$ is a transversal of $\pi(\alpha_{i+1})$ ($k$-block transitivity of $G$ insures their existence). Consider all the products of the form $\alpha_1, \alpha_2 \alpha_1, \alpha_3 \alpha_2 \alpha_1, \ldots$. Since $\langle \alpha : G \rangle$ is finite there exist integers $m < j$ such that $\alpha_2 \alpha_{j-1} \ldots \alpha_{m+1} \alpha_m \ldots \alpha_1 = \alpha_2 \alpha_3 \alpha_1, \ldots$. Let $\delta = \alpha_j \ldots \alpha_{m+1}$ and $\gamma = \alpha_m \ldots \alpha_1$. Then $\delta \gamma = \gamma$ so $\text{im} \delta \supseteq \text{im} \gamma$, and since $|\text{im} \delta| = |\text{im} \alpha| = |\text{im} \gamma|$ we have that $\text{im} \delta = \text{im} \gamma$. Thus $\delta$ is the identity on its image, and so $\delta$ is an idempotent having $\text{im} \delta = \text{im} \alpha_1$ and $\pi(\delta) = \pi(\alpha_{m+1})$. Let $h \in G$ be such that $\alpha_{m+1} = h \alpha h^{-1}$, then $\beta = h^{-1} \delta h$ is the required idempotent. Indeed $\beta^2 = h^{-1} \delta hh^{-1} \delta h = h^{-1} \delta^2 h = h^{-1} \delta h = \beta$ and $\pi(\beta) = \pi(h^{-1} \delta h) = h^{-1}(\pi(\delta)) = h^{-1}(\pi(\alpha_{m+1})) = h^{-1}(\pi(h \alpha h^{-1})) = h^{-1}(\pi(h(\alpha))) = \pi(\alpha)$.

Since Alt$_n$ is $k$-block-transitive for any $1 \leq k \leq n - 1$ we have the following.

Corollary 4. $\langle \alpha : \text{Alt}_n \rangle$ contains an idempotent $\beta$ with $\pi(\beta) = \pi(\alpha)$.

We say that $\alpha$ has a partition of type $1^{k_1}2^{k_2} \ldots k_r$ if $\pi(\alpha)$ has $k_i$ classes of size $i$. \[ \square \]
Finite Transformation Semigroups 29

Lemma 5. Let \( \alpha \in T_n - S_n \) be an idempotent. There exists an \( h \in S_n - \text{Alt}_n \) such that \( h\alpha h^{-1} \in \langle \alpha : \text{Alt}_n \rangle \), \( n \geq 3 \).

Proof. Assume that there exist \( x, y \in \text{im} \alpha \) such that \( \alpha^{-1}(x) = \{x\} \) and \( \alpha^{-1}(y) = \{y\} \). Then for the transposition \( h = (x, y) \) we have \( h\alpha h^{-1} = \alpha \in \langle \alpha : \text{Alt}_n \rangle \). Now suppose \( \pi(\alpha) \) contains a class \( A \) having \( |A| \geq 3 \). Let \( a, b \in A - \text{im} \alpha \). Then for \( h = (a, b) \) we have \( h\alpha h^{-1} = \alpha \in \langle \alpha : \text{Alt}_n \rangle \).

If none of the above holds then \( \alpha \) has a partition of type \( 1^0 2^k = 2^k \) \( (k = n/2, n \) is even) or \( 1^1 2^k \) \( (k = (n-1)/2, n \) is odd) \( \). Let \( \alpha_1, \alpha_2 \) be idempotents in \( T_{2k} \) and \( T_{2k+1} \) respectively, \( \alpha_1 = [1, 1, 3, 3, \ldots, 2k-1, 2k-1] \) and \( \alpha_2 = [1, 1, 3, 3, \ldots, 2k-1, 2k-1, 2k+1] \) \( \). Write \( [\alpha_1, \alpha_2, \ldots, \alpha_l] \) for a transformation mapping \( i \) to \( \alpha_i \). We may assume without loss of generality that \( \alpha \) equals to either \( \alpha_1 \) or \( \alpha_2 \). It is easy to verify that for \( h = (12) \) and \( n \geq 5 \) we have

\[
h\alpha_1 h^{-1} = (12)(35) \alpha_1 (35)(12) \alpha_1 \in \langle \alpha : \text{Alt}_n \rangle.
\]

If \( n = 4 \), then \( \alpha_1 = [1, 1, 3, 3] \), and for \( h = (12) \),

\[
h\alpha_1 h^{-1} = ((132)\alpha_1 (123))(134)\alpha_1 (143) \alpha_1 \in \langle \alpha_1 : \text{Alt}_4 \rangle.
\]

If \( n = 3 \), \( \alpha_2 = [1, 1, 3] \), and for \( h = (12) \),

\[
h\alpha_2 h^{-1} = ((132)\alpha_2 (123))(123)\alpha_2 (132)\alpha_2 (132) \alpha_2 \in \langle \alpha_2 : \text{Alt}_3 \rangle.
\]

Proposition 6. Let \( \alpha \in T_n \), \( n \geq 3 \). Then \( \langle \alpha : S_n \rangle \).

Proof. Observe that we only need to show that \( \langle \alpha : S_n \rangle \subseteq \langle \alpha : \text{Alt}_n \rangle \). If \( \alpha \in \text{Alt}_n \) then \( \langle \alpha : S_n \rangle \subseteq \text{Alt}_n \subseteq S_n \). Also \( \langle \alpha : \text{Alt}_n \rangle \subseteq \text{Alt}_n \), and since \( \text{Alt}_n \) is simple for \( n \neq 4 \) \( [4, 10.8.7] \) we have that \( \langle \alpha : \text{Alt}_n \rangle = \text{Alt}_n \) if \( \alpha \neq (1) \) \( \) and \( \langle (1) : \text{Alt}_n \rangle = \{1\} \) \( \) (provided \( n \neq 4 \)). If \( n = 4 \), \( \alpha \neq (1) \) \( \) and \( \langle \alpha : \text{Alt}_4 \rangle \neq \text{Alt}_4 \) then \( \langle \alpha : \text{Alt}_4 \rangle = V \), the 4-group, \( \) and \( \alpha \in V \). Since \( V \subseteq S_4 \), \( \langle \alpha : S_4 \rangle \subseteq V \) \( \) also, and therefore \( \langle \alpha : S_4 \rangle \subseteq \langle \alpha : \text{Alt}_4 \rangle \), as required.

If \( \alpha \) is an odd permutation then for any \( q \in S_n - \text{Alt}_n \), \( q = \sigma(\alpha^{-1} q) \), \( \alpha^{-1} q \in \text{Alt}_n \), \( \) and \( q \sigma q^{-1} = \sigma(\alpha^{-1} q) \sigma(\alpha^{-1} q)^{-1} = \alpha^{-1} q \alpha^{-1} q^{-1} \in \langle \alpha : \text{Alt}_n \rangle \), \( \) so \( \langle \alpha : S_n \rangle \subseteq \langle \alpha : \text{Alt}_n \rangle \) again.

Now let \( \alpha \in T_n - S_n \). By Corollary 4, \( \langle \alpha : \text{Alt}_n \rangle \) contains an idempotent \( \beta \) with \( \pi(\beta) = \pi(\alpha) \). By Lemma 5 and Corollary 2, \( \langle \alpha : \text{Alt}_n \rangle \supseteq \langle \beta : \text{Alt}_n \rangle = \langle \beta : S_n \rangle = \langle \alpha : S_n \rangle \) \( \) (for a transformation \( \gamma \) the semigroup \( \langle \gamma : S_n \rangle \) comprises all \( \delta \in T_n \) having \( \pi(\delta) \supseteq \), a partition of the same type as \( \pi(\gamma) \) \( \) [2]).

Corollary 7. There is no \( S \subseteq T_n \) such that \( G_S = \text{Alt}_n \).

Proof. Suppose \( \text{Alt}_n \subseteq G_S \). Then by Proposition 6, for any \( \alpha \in S, h \in S_n \), we have that \( h\alpha h^{-1} \in \langle \alpha : S_n \rangle \) \( \Rightarrow \) \( \langle \alpha : \text{Alt}_n \rangle \subseteq S \), that is \( h \in G_S \) \( \) and \( G_S = S_n \).
Now to prove our main Theorem suppose that $G \subseteq \text{Inn} S$ such that $G \cong \text{Alt}_n$. Let $\bar{G} = \{h \in S_n: \phi_h \in G\}$. Then $\bar{G} \leq G_S \leq S_n$, and the order of $\bar{G}$ is at least that of $\text{Alt}_n$. Therefore $\bar{G}$ contains $\text{Alt}_n$, and by Corollary 7, $G_S = S_n$.

We note that the above result is not necessarily true for semigroups of transformations of infinite sets. For example, let $X = \mathbb{Z}$ be the set of all integers and $\alpha: \mathbb{Z} \to \mathbb{Z}$ such that $\alpha(a) = 2a$, for all $a \in \mathbb{Z}$. Let $S_Z$ be the symmetric group on $\mathbb{Z}$. The alternating subgroup $\text{Alt}_Z$ of $S_Z$ consists of all the finite even permutations of $\mathbb{Z}$. Then $\langle \alpha: S_Z \rangle = \{\beta: \mathbb{Z} \to \mathbb{Z} \mid \beta \text{ is } 1-1 \text{ and } |\mathbb{Z} - \text{im } \beta| = \aleph_0\}$. In particular $\langle \alpha: S_Z \rangle$ contains $\beta$ defined by $\beta(a) = 2a - 1$ for all $a \in \mathbb{Z}$. Observe that for all $a \in \mathbb{Z}$, $\alpha(a) \neq \beta(a)$. Since any $h \in \text{Alt}_Z$ moves at most a finite number of points, $\beta \notin \langle \alpha: \text{Alt}_Z \rangle$.

For a transformation $\alpha$ of $X$ let shift $\alpha = |\{x \in X : \alpha(x) \neq x\}|$. Let $\nu$ be an infinite cardinal not exceeding $|X|^+$, the cardinal successor of $|X|$, and let $\text{Sym}(X, \nu)$ be the subgroup of all permutations in $S_X$ whose shift is less than $\nu$.

Conjecture 1. If shift $\alpha = u$ then $\langle \alpha: \text{Sym}(X, w) \rangle = \langle \alpha: S_X \rangle$ for all $w \geq u^+$.

2. There is no semigroup $S$ of transformations of $X$ having $G_S = \text{Sym}(X, |X|)$.

Observe that permutations $h$ and $p$ in $G_S$ give rise to equal automorphisms $\phi_h$ and $\phi_p$ if and only if $h^{-1}p$ is in the centralizer $C(S)$ of $S$, $C(S) = \{\alpha \in T_n : \alpha \beta = \beta \alpha \text{ for all } \beta \in S\}$. Thus $G_S$ is isomorphic to the group $\text{Inn} S$ of the inner automorphisms of $S$ if and only if $C(S) \cap G_S$ consists of the identity permutation. The results of this paper in conjunction with the above observations give rise to the following.

Problem 1. Characterize these subgroups $G$ of $S_n$ having $G = G_S$ for some subsemigroup $S$ of $S_n$.

2. Given that $G = G_T$ for some $T \subseteq T_n$ characterize all $S \subseteq T_n$ such that $G_S = G$.

3. Characterize these subsemigroups $S$ of $T_n$ having $|C(S) \cap G_S| = 1$.

REFERENCES


MATHEMATICS DEPARTMENT
UNIVERSITY OF LOUISVILLE
LOUISVILLE, KY 40292
U.S.A.