DIRECT PATHS OF WAVELETS

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ABSTRACT. We associate a von Neumann algebra with each pair of complete wandering vectors for a unitary system. When this algebra is nonatomic, there is a norm–continuous path of a simple nature connecting the original pair of wandering vectors. We apply this technique to wavelet theory and compute the above von Neumann algebra in some special cases. Results from selection theory and ergodic theory lead to nontrivial examples where both atomic and nonatomic von Neumann algebras occur.

1. INTRODUCTION

We consider a particularly simple type of continuous path of orthonormal (mother) wavelets that may connect two given wavelets. The general connectedness problem for wavelets has been studied by several authors over the past decade, and there is a literature concerning it. The solution is known to be yes (i.e. they are path-connected in the $L^2(\mathbb{R})$-norm) for the special case of the dyadic MRA (multiresolution analysis) wavelets on $\mathbb{R}$ (by the Wutam Consortium [27]), and for the case of the MSF-wavelets, or equivalently the wavelet measurable sets (by Speegle [26]). The general problem for arbitrary orthonormal wavelets remains open. This article is in an essentially different (but related) direction. The difference is that we focus on a special type of continuous path between wavelets, rather than consider the existence of generic paths.

The primary motivation for this paper is the development of the mathematics underlying the subject of wavelet theory in Hilbert space. New connectivity

results can be used for classification purposes in that two wavelets can be considered to be related in a particular way if there is a particular type of continuous path of wavelets connecting them. In addition, new results for continuous paths of wavelets have potential use in development of perturbation techniques for wavelet analysis. We were led to this paper by an attempt to gain a better understanding of the known positive results in [27] and [26] mentioned above, and its relationship with our work in [2] and [15]. We discovered that certain (but not all) pairs of wavelets can be connected by a direct path (see definition below) analogous to the manner in which two measurable sets of the same measure can be connected in the symmetric difference metric by continuously displacing a continuously increasing portion of the first set by an equivalent portion of the second set, fixing the intersection. This concept is natural within the class of wavelet sets and we discovered that it can be modeled and analyzed very effectively using operator algebra techniques, and generalized appropriately, leading to our definition. Examples abound, and there are simple examples of wavelets which are connected but not directly connected. Direct connection is quite special. However, using some earlier results of the authors leading up to this paper, many pairs of wavelet sets, hence MSF-wavelets, can be shown to be directly connected. Speegle's interesting paths constructed in [26], which prove connectivity of all wavelet sets, are not direct paths. The direct connectivity of arbitrary pairs of wavelet sets still remains an open problem.

Following [3], a *unitary system* $\mathcal{U}$ acting on a Hilbert space H is a collection of unitary operators in $\mathcal{B}(H)$ containing I. A *complete wandering vector* for $\mathcal{U}$, if one exists, is a vector $\psi \in H$ with the property that $\mathcal{U}\psi$ is an orthonormal basis for H. Let $\mathcal{W}(\mathcal{U})$ denote the set of all complete wandering vectors for $\mathcal{U}$. If $\mathcal{S} \subset \mathcal{B}(H)$ is a set of operators and $\mathcal{E} \subset H$ is a set of vectors, define the $\mathcal{E}$-*commutant* of $\mathcal{S}$ to be

$$\mathcal{C}_{\mathcal{E}}(\mathcal{S}) := \{A \in \mathcal{B}(H) : (AS - SA)\mathcal{E} = \{0\}, \text{ for all } S \in \mathcal{S}\}.$$

This is a W.O.T. closed linear subspace of $\mathcal{B}(H)$. If $x \in H$, define the *local commutant of $\mathcal{S}$ at x* to be $\mathcal{C}_x(\mathcal{S}) := \mathcal{C}_{\{x\}}(\mathcal{S}) = \{A \in \mathcal{B}(H) : (AS - SA)x = 0, \text{ for all } S \in \mathcal{S}\}$. Then $\mathcal{C}_{\mathcal{E}}(\mathcal{S}) = \underset{x \in \mathcal{E}}{\cap} \mathcal{C}_x(\mathcal{S})$. (It follows that $\mathcal{C}_{\mathcal{E}}(\mathcal{S})$ is 2-reflexive, see ([1]).) An easy argument shows that if x is cyclic for $\mathcal{U}$, in the sense that $\underset{U \in \mathcal{U}}{\vee} Ux = H$, then x is separating for $\mathcal{C}_x(\mathcal{U})$.

If $\psi, \eta \in \mathcal{W}(\mathcal{U})$, let V_ψ^η denote the unitary operator defined by

$$V_\psi^\eta(\sum_{U \in \mathcal{U}} c_U U\psi) = \sum_{U \in \mathcal{U}} c_U U\eta, \quad \{c_U\} \in \ell^2(\mathcal{U}).$$

Then, an easy computation shows that $V_\psi^\eta \in \mathcal{C}_\psi(\mathcal{U})$. Reversing this argument, one can show that if V is any unitary operator in $\mathcal{C}_\psi(\mathcal{U})$ then $V\psi \in \mathcal{W}(\mathcal{U})$. Since ψ separates $\mathcal{C}_\psi(\mathcal{U})$, it follows that V_ψ^η is the unique unitary operator in $\mathcal{C}_\psi(\mathcal{U})$ that maps ψ to η.

If $\psi, \eta \in \mathcal{W}(\mathcal{U})$, let us say that $\{\psi, \eta\}$ is a *nonatomic pair* of complete wandering vectors for $\mathcal{U}$ if $\{\mathcal{U}, V_\psi^\eta\}'$ (the operators commuting with V_ψ^η and all the unitaries in $\mathcal{U}$) is a *nonatomic* von Neumann algebra (i.e. it contains no nonzero minimal projections). In general, we will call $\{\mathcal{U}, V_\psi^\eta\}'$ the *essential von Neumann algebra* for the pair $\{\psi, \eta\}$ and write

$$\mathcal{M}_{\psi,\eta} := \{\mathcal{U}, V_\psi^\eta\}'.$$

A standard argument using Fuglede's Theorem shows that $\mathcal{M}_{\psi,\eta}$ is a von Neumann algebra. Since $V_\psi^\eta = (V_\eta^\psi)^*$ we have $\mathcal{M}_{\psi,\eta} = \mathcal{M}_{\eta,\psi}$.

We have the following simple fact.

Proposition 1.1. *If $\{\psi, \eta\}$ is a nonatomic pair in $\mathcal{W}(\mathcal{U})$, then there exists a continuous path $\alpha(t)_{t \in [0,1]}$ in $\mathcal{W}(\mathcal{U})$ with $\alpha(0) = \psi$ and $\alpha(1) = \eta$.*

PROOF. A standard argument implies that $\mathcal{M}_{\psi,\eta}$ contains a nest $\{P_t : 0 \le t \le 1\}$ of projections with $P_0 = 0$, $P_1 = I$, such that $t \to P_t$ is (strong-operator-topology)-continuous. Let

(1) $$\alpha(t) := (I - P_t)\psi + P_t\eta, \quad 0 \le t \le 1.$$

Then $\alpha(t)$ is a norm-continuous path of vectors in H. Let $W_t := I - P_t + P_t V_\psi^\eta$, $0 \le t \le 1$. Clearly $\alpha(t) = W_t\psi$. Since $P_t \in \mathcal{U}'$ and $\mathcal{C}_\psi(\mathcal{U})$ is closed under left multiplication by elements of $\mathcal{U}'$ (notice that $V_\psi^\eta, I \in \mathcal{C}_\psi(\mathcal{U})$), we have $W_t \in \mathcal{C}_\psi(\mathcal{U})$ for all $t \in [0,1]$. Moreover since P_t commutes with V_ψ^η, one can easily check that W_t is unitary. So, $\alpha(t) \in \mathcal{W}(\mathcal{U})$ for all t. $\square$

We will call the path (1) connecting the elements of a nonatomic pair a *direct path* of wavelets, and we will say that the wavelets in a nonatomic pair are *directly connected*.

2. ORTHONORMAL DYADIC WAVELETS

For the reader convenience let us introduce the notation and terminology of the classical case of dyadic orthonormal one-dimensional wavelets. For this setting we let H be the L^2-space with respect to μ (Lebesgue measure) on $\mathbb{R}$ which is denoted as usual by $L^2(\mathbb{R})$. The unitary system $\mathcal{U}$ is determined by the unitary operators D and T (bilateral shifts of infinite multiplicity) defined on $L^2(\mathbb{R})$ by

$$(Df)(x) = \sqrt{2}f(2x) \quad \text{and} \quad (Tf)(x) = f(x-1), \ a.e. \ x \in \mathbb{R}, \ f \in L^2(\mathbb{R}),$$

in the following way:

$$\mathcal{U}\, (=\mathcal{U}_{D,T}) := \{D^k T^l : k, l \in \mathbb{Z}\}.$$

A function $w \in L^2(\mathbb{R})$ is called simply a *wavelet* if $w \in \mathcal{W}(\mathcal{U}_{D,T})$. It is well-known (see, e.g. [13]) that $\widehat{w}$ is bounded if w is a wavelet. We say that a subset G of $\mathbb{R}$ of positive measure is a *wavelet set* if $\frac{1}{\sqrt{2\pi}}\chi_G = \widehat{w}$, where w is a wavelet in $L^2(\mathbb{R})$ and $\widehat{f}$ denotes the Fourier-Plancherel transform of the function f, which for functions in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is defined by

$$\widehat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-itx} f(t) d\mu(t), \quad x \in \mathbb{R}.$$

The inverse Fourier transform of a function $f \in L^2(\mathbb{R})$ is denoted by $\check{f}$. A wavelet w satisfying $\frac{1}{\sqrt{2\pi}}\chi_G = |\widehat{w}|$ for some wavelet set G is called a *(MSF)-wavelet*, and we call G the support of $\widehat{w}$. One knows (see, e.g. [3]) that the set of exponentials $\{e^{inx}|_G : n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(G)$.

One of the simplest examples of such sets is the Littlewood-Paley wavelet set $E := [-2\pi, -\pi) \cup [\pi, 2\pi)$. One less obvious example of a wavelet set is the following union of eight intervals

$$
\begin{aligned}
(2) \qquad S := \quad & \left[-\frac{4\pi}{3}, -\frac{5\pi}{4}\right) \cup \left[-\pi, -\frac{2\pi}{3}\right) \cup \left[-\frac{5\pi}{8}, -\frac{\pi}{2}\right) \cup \left[\frac{4\pi}{7}, \frac{2\pi}{3}\right) \\
& \cup \left[\frac{3\pi}{4}, \pi\right) \cup \left[\frac{4\pi}{3}, \frac{11\pi}{8}\right) \cup \left[4\pi, \frac{32\pi}{7}\right) \cup \left[\frac{11\pi}{2}, 6\pi\right),
\end{aligned}
$$

which was constructed in [14]. The wavelet corresponding to the Littlewood-Paley wavelet set (via the Fourier transform) is called the Shannon wavelet.

For an arbitrary wavelet set F it was proved in [15] that there exists another wavelet set $\widetilde{F}$ (called *regularized*) with the properties that $F = \widetilde{F}$ a.e. $[\mu]$ and such that $\mathbb{R} = \bigcup_{k \in \mathbb{Z}} (\widetilde{F} + 2k\pi)$ (disjoint union) and $\mathbb{R} \setminus \{0\} = \bigcup_{k \in \mathbb{Z}} (2^k \widetilde{F})$(disjoint union). Let us denote by $\mathcal{WS}$ the class of all regularized wavelet sets.

Hence, for $F \in \mathcal{WS}$ we can define the following two maps associated with F: let $\tau_F : \mathbb{R} \to F$ be the function defined by $\tau_F(x) = x + 2j\pi$, where j is the unique integer satisfying $x + 2j\pi \in F$ and let $\delta_F : \mathbb{R}\backslash\{0\} \to F$ be the map defined by $\delta_F(x) = 2^k x$, where k is the unique integer for which $2^k x \in F$. If G is another regularized wavelet set it is clear that the restrictions of τ_F and δ_F to G are measurable bijections. Thus we can associate with every F, $G \in \mathcal{WS}$ (F, G regularized) a measurable bijection on F [resp. G] defined by

$$(3) \qquad h_{FG} := \tau_{F|G} \circ \delta_{F|G}^{-1}, \quad [resp. \ h_{GF} := \tau_{G|F} \circ \delta_{G|F}^{-1}].$$

In case $F := E$ (Littlewood-Paley wavelet set) we simply denote h_{EG} by h_G.

It turns out that the conjugation $\widetilde{h}_G := \xi \circ h_G \circ \xi^{-1} : [0,1) \to [0,1)$ of h_G, by the function $\xi : E \to [0,1)$ defined by

$$(4) \qquad \xi(x) = \begin{cases} \dfrac{x}{2\pi}, & x \in [\pi, 2\pi) \\[2mm] \dfrac{x}{2\pi} + 1, & x \in [-2\pi, -\pi), \end{cases}$$

takes a simpler form than h_G. The following characterization for the class of wavelet sets in terms of the corresponding maps $\widetilde{h}_G$ has been established in in ([14]). We shall term the maps $\widetilde{h}_G$ *wavelet induced* maps.

Proposition 2.1. ([14]) *Let $G \in \mathcal{WS}$ and $\widetilde{h}_G$ be defined as above. Then the map $\widetilde{h}_G$ has the following properties:*

(i) *$\widetilde{h}_G$ is a measurable bijection of $[0,1)$,*
(ii) *there exists a measurable partition $\{A_k\}_{k \in \mathbb{Z}}$ of $[\frac{1}{2}, 1)$ and a measurable partition $\{B_k\}_{k \in \mathbb{Z}}$ of $[0, \frac{1}{2})$, such that*

$$(5) \qquad \widetilde{h}_G(x) = \begin{cases} \lfloor 2^k x \rfloor, & x \in A_k, \ k \in \mathbb{Z}, \\[2mm] \lfloor 2^k(x-1) \rfloor, & x \in B_k, \ k \in \mathbb{Z}, \end{cases}$$

where $\lfloor x \rfloor$ denotes the fractional part of the real number x.

Moreover, if h is a map satisfying (i) and (ii) then there exists a (regularized) wavelet set G such that $h = \widetilde{h}_G$ (a.e. $[d\mu]$).

For the wavelet set S given by (2) for instance, one can compute $\widetilde{h}_S$ and obtain the following wavelet induced map

$$
(6) \qquad \widetilde{h}_S(x) =
\begin{cases}
\lfloor 2^{-1}(x-1)\rfloor & \text{on} \quad \left[0,\dfrac{1}{3}\right)\cup\left[\dfrac{3}{8},\dfrac{1}{2}\right), \\[2mm]
\lfloor 2^2 x\rfloor & \text{on} \quad \left[\dfrac{1}{2},\dfrac{4}{7}\right)\cup\left[\dfrac{11}{16},\dfrac{3}{4}\right), \\[2mm]
\lfloor 2^{-1} x\rfloor & \text{on} \quad \left[\dfrac{4}{7},\dfrac{2}{3}\right)\cup\left[\dfrac{3}{4},1\right), \\[2mm]
\lfloor x\rfloor & \text{on} \quad \left[\dfrac{1}{3},\dfrac{3}{8}\right)\cup\left[\dfrac{2}{3},\dfrac{11}{16}\right).
\end{cases}
$$

We denote the class of all wavelet induced maps by $\mathcal{WI}$. By Proposition 2.1, there exists a one-to-one correspondence between $\mathcal{WS}$ and $\mathcal{WI}$.

Before we state our results about the essential algebra for a pair of wavelets we introduce more terminology and point out some simple observations.

Let D_a $(a>0)$ and T_b $(b\in\mathbb{R})$ denote the unitary operators on $L^2(\mathbb{R})$ defined by $(D_a f)(x)=\sqrt{a}f(ax)$, $x\in\mathbb{R}$, $f\in L^2(\mathbb{R})$ and $(T_b f)(x)=x-b$, $x\in\mathbb{R}$, $f\in L^2(\mathbb{R})$. It is easy to check that we have the following relations satisfied by these operators:

(7) $T_b D_a = D_a T_{ab}, \quad D_a D_a' = D_{aa'}, \quad T_b T_b = T_{b+b'}, \quad a,a'\in(0,\infty), \ b,b'\in\mathbb{R}.$

For $A\in\mathcal{B}(L^2(\mathbb{R}))$ we write $\widehat{A}$ for the bounded linear operator defined by $\widehat{A}\widehat{f}=\widehat{Af}$, $f\in L^2(\mathbb{R})$. If $\mathcal{A}\subset\mathcal{B}(L^2(\mathbb{R}))$ then $\widehat{\mathcal{A}}:=\{\widehat{A}:A\in\mathcal{A}\}$. It is easy to check that $(\widehat{D}f)(x)=\frac{1}{\sqrt{2}}f(\frac{x}{2})$, $x\in\mathbb{R}$, $f\in L^2(\mathbb{R})$, and $(\widehat{T}f)(x)=e^{-ix}f(x)$, $x\in\mathbb{R}$, $f\in L^2(\mathbb{R})$. In [3] it was proved that

(8) $\{\widehat{D},\widehat{T}\}'(=\mathcal{DP}):=\{M_f:\quad f\in L^\infty(\mathbb{R}),\quad f(s)=f(2s)\quad a.e.\quad [\mu],\ s\in\mathbb{R}\},$

(M_f is the operator of multiplication by f on $L^2(\mathbb{R})$). $\mathcal{DP}$ is called the *dilation-periodic-algebra*. Hence, in the classical case $\widehat{\mathcal{M}_{\psi,\eta}}$ is just a subalgebra of $\mathcal{DP}$ which, in particular, implies that $\mathcal{M}_{\psi,\eta}$ is an abelian von Neumann algebra for every $\psi,\eta\in\mathcal{W}(\mathcal{U})$. For an arbitrary $F\in\mathcal{WS}$, it is clear from (8) that the algebra $\mathcal{DP}$ is isomorphic to $L^\infty(F)$, via the isomorphism $f\to M_{\widetilde{f}}$, $f\in L^\infty(F)$, where

(9) $$\widetilde{f}(x) = f(\delta_F(x)) \quad a.e. \quad [\mu], \quad x\in\mathbb{R}.$$

Proposition 2.2. *Let $\psi,\eta\in\mathcal{W}(\mathcal{U})$, $A\in\mathcal{B}(L^2(\mathbb{R}))$ such that A commutes with D and T. Let us assume that the representation of $A\psi$ in the basis $\{D^k T^l\psi\}_{k,l\in\mathbb{Z}}$*

is given by:

$$(10) \qquad A\psi = \sum_{s,t\in\mathbb{Z}} a_{s,t} D^s T^t \psi, \quad (\sum_{s,t\in\mathbb{Z}} |a_{s,t}|^2 < \infty).$$

Then the following are equivalent:

(i) $A \in \mathcal{M}_{\psi,\eta}$,

(ii) $A(D^l T^m \eta) = \sum_{s,t\in\mathbb{Z}} a_{s,t} V_\psi^\eta D^{l+s} T_{t+2^s m} \psi$ *for every* $l, m \in \mathbb{Z}$,

(iii) *(under the assumption that ψ is a (MSF)-wavelet)*

$$(11) \qquad A(D^k T^l \eta) = \sum_{m\in\mathbb{Z}} a_{0,m} D^k T^{l+m} \eta, \quad \text{for every } k, l \in \mathbb{Z}.$$

PROOF. Let $A \in \mathcal{B}(L^2(\mathbb{R}))$ an operator commuting with and D and T. Then $A \in \mathcal{M}_{\psi,\eta}$ if and only if A commutes with V_ψ^η. The equation $AV_\psi^\eta = V_\psi^\eta A$ is equivalent to $AV_\psi^\eta(D^l T^m \psi) = V_\psi^\eta A(D^l T^m \psi)$ for every $l, m \in \mathbb{Z}$. Since A commutes with D and T we have equivalently:

$$(12) \qquad A(D^l T^m \eta) = V_\psi^\eta D^l T^m (A\psi), \quad l, m \in \mathbb{Z}.$$

Using the representation of $A\psi$ in (10) the relations in (12) completely determine A on the elements of the basis $\{D^l T^m \eta\}_{l,m\in\mathbb{Z}}$:

$$A(D^l T^m \eta) = \sum_{s,t\in\mathbb{Z}} a_{s,t} V_\psi^\eta D^l T^m D^s T^t \psi, \quad l, m \in \mathbb{Z}.$$

Therefore, taking into account that $T^s = T_s$, $D^s = D_{2^s}$ for all $s \in \mathbb{Z}$ and the relations (7), the condition $AV_\psi^\eta = V_\psi^\eta A$ becomes equivalent to

$$(13)$$
$$A(D^l T^m \eta) = \sum_{s,t\in\mathbb{Z}} a_{s,t} V_\psi^\eta D^{l+s} T_{t+2^s m} \psi, \quad l,m\in\mathbb{Z},$$

or

$$A(D^l T^m \eta) = \sum_{s,t\in\mathbb{Z},s\geq 0} a_{s,t} D^{l+s} T_{t+2^s m} \eta + \sum_{s,t\in\mathbb{Z},s<0} a_{s,t} V_\psi^\eta D^{l+s} T_{t+2^s m} \psi, \quad l,m\in\mathbb{Z},$$

This shows the equivalence of (i) with (ii).

Remark. If ψ is a (MSF)-wavelet it turns out that the coefficients $a_{s,t}$ of the second sum in (13) are zero and these equations become considerably simpler.

Indeed, if ψ is a (MSF)-wavelet then $\psi = \frac{1}{\sqrt{2\pi}} \check{g}$ for some measurable function g of modulus one supported on $F \in \mathcal{WS}$. Since $A \in \{D, T\}'$ we may assume

that $\widehat{A} = M_f$ with $f \in L^\infty(\mathbb{R})$. Hence $\widehat{A}\widehat{\psi} = M_f \frac{1}{\sqrt{2\pi}} g = \frac{1}{\sqrt{2\pi}} f_{|F} g$. Now, since $\{e_m(x) = \frac{1}{\sqrt{2\pi}} e^{imx}\}_{m \in \mathbb{Z}}$ is an orthonormal basis for $L^2(F)$, we have a representation of $f_{|F}$ in this basis, say $f(x) = \sum_{m \in \mathbb{Z}} c_m e^{imx}$, $x \in F$, with convergence in $L^2(F)$. Then, $\widehat{A}\widehat{\psi}(x) = \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} c_m e^{imx} g(x) = \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} c_m (\widehat{T}^{-m} g)(x)$, with convergence in $L^2(F)$ which is the same, in this case, as convergence in $L^2(\mathbb{R})$. Thus, $A\psi = \sum_{m \in \mathbb{Z}} c_m T^{-m} \psi$, which implies that $a_{0,m} = c_{-m}$ and $a_{k,m} = 0$ for all $k \neq 0$, $k, m \in \mathbb{Z}$. The condition on A to commute with V_ψ^η (equalities in (13)) is then equivalent to the condition (11) in (iii). $\qquad \square$

Theorem 2.3. *Let ψ be a MSF-wavelet supported on $F \in \mathcal{WS}$ and let $\eta \in \mathcal{W}(\mathcal{U})$. Then $\mathcal{M}_{\psi,\eta}$ is isomorphic (via the Fourier transform) to the subalgebra of $\mathcal{DP}$*

$$(14) \qquad \{M_f \in \mathcal{DP} : \quad (f - f \circ \tau_F)\widehat{\eta} = 0\}.$$

PROOF. By Proposition 2.2, it suffices to analyze how (11) changes when we apply the Fourier transform to it. For $A \in \mathcal{M}_{\psi,\eta}$ there is $f \in L^\infty(\mathbb{R})$ such that $f(s) = f(2s)$ a.e. $s \in \mathbb{R}$, and $\widehat{A} = M_f$. Then (11) becomes:

$$2^{-\frac{k}{2}} f(x) e^{\frac{-ilx}{2^k}} \widehat{\eta}\left(\frac{x}{2^k}\right) = \sum_{m \in \mathbb{Z}} c_m 2^{-\frac{k}{2}} e^{\frac{-i(l-m)x}{2^k}} \widehat{\eta}\left(\frac{x}{2^k}\right), \quad k, l \in \mathbb{Z}, \text{ in } L^2(\mathbb{R}),$$

where, for the convenience of the reader, we recall that $c_m = \frac{1}{\sqrt{2\pi}} < f|_F, e_m >_{L^2(F)}$, $m \in \mathbb{Z}$.

After obvious simplifications and the change of variable $\frac{x}{2^k} = y$ we obtain that the above equality is equivalent to

$$\left(f(2^k y) - \sum_{m \in \mathbb{Z}} c_m e^{imy}\right) \widehat{\eta}(y) = 0 \text{ with convergence in } L^2(\mathbb{R}).$$

Taking into account that f is 2-dilation periodic the last equality together with the above remarks imply the following equivalent condition on f:

$$(15) \qquad f(x) = \sum_{m \in \mathbb{Z}} c_m e^{imx} \quad \text{in} \quad L^2(\mathbb{R}, |\widehat{\eta}|^2 d\mu).$$

Let us first show now that the relation in (14) follows from (15). Indeed, using (15), we obtain that for some increasing sequence of integers $\{n_k\}_{k \in \mathbb{N}}$ we have

$$\lim_{n_k \to \infty} \left(\widetilde{f}(x) - \sum_{m=-n_k}^{m=n_k} c_m e^{imx}\right) \widehat{\eta}(x) = 0 \text{ a.e. } x \in \mathbb{R}.$$

But the sequence of functions $\{\sum_{l=-n_k}^{l=n_k} c_l e^{ilx}\}_{k \in \mathbb{N}}$ contains a subsequence which converges a.e. on $\mathbb{R}$ to $f(\tau_F(x))$ and so the above equation implies that $(f(x) - f(\tau(x)))\widehat{\eta}(x) = 0$ for a.e. $x \in \mathbb{R}$.

Next, we show that the condition on f in (14) implies (15). Since $(f(x) - f(\tau(x)))\widehat{\eta}(x) = 0$ a.e. on $\mathbb{R}$, we write

$$\int_{\mathbb{R}} \left| f(x) - \sum_{m=-k}^{m=n} c_m e^{imx} \right|^2 |\widehat{\eta}(x)|^2 d\mu(x) = \int_{\mathbb{R}} \left| f(\tau(x)) - \sum_{m=-k}^{m=n} c_m e^{imx} \right|^2 |\widehat{\eta}(x)|^2 d\mu(x)$$

Since η is a wavelet one can show that $\sum_{l \in \mathbb{Z}} |\widehat{\eta}(x + 2l\pi)|^2 = 2\pi$ for a.e. $x \in \mathbb{R}$ (see for instance [13]). Hence we may continue with the right hand side of the above equality as follows:

$$\int_{\mathbb{R}} \left| f(\tau(x)) - \sum_{m=-k}^{m=n} c_m e^{imx} \right|^2 |\widehat{\eta}(x)|^2 d\mu(x)$$

$$= \sum_{l \in \mathbb{Z}} \int_{F-2l\pi} \left| f(\tau(x)) - \sum_{m=-k}^{m=n} c_m e^{imx} \right|^2 |\widehat{\eta}(x)|^2 d\mu(x)$$

$$= \sum_{l \in \mathbb{Z}} \int_F \left| f(x) - \sum_{m=-k}^{m=n} c_m e^{imx} \right|^2 |\widehat{\eta}(x + 2l\pi)|^2 d\mu(x)$$

$$= 2\pi \int_F \left| f(x) - \sum_{m=-k}^{m=n} c_m e^{imx} \right|^2 d\mu(x) \xrightarrow[\text{as } k,n \to \infty]{} 0 \ .$$

This concludes our proof. $\qquad\square$

Corollary 2.4. *Suppose ψ is a MSF-wavelet supported on $F \in \mathcal{WS}$ and $\eta \in \mathcal{W}(\mathcal{U})$ is supported on $G \in \mathcal{WS}$. Then $\mathcal{M}_{\psi,\eta}$ is isomorphic to the subalgebra of $L^\infty(F)$ [resp. $L^\infty(G)$]*

(16) $\quad \{f \in L^\infty(F): \quad f = f \circ h_{FG}\}, \quad [resp.\{f \in L^\infty(G): \quad f = f \circ h_{GF}\}]$

where h_{FG}[resp. h_{GF}] is defined by (3).

PROOF. Any 2-dilation periodic function on $\mathbb{R}$ can be regarded as a function $f \in L^\infty(F)$ extended as in (9) simply by setting $f(x) := f(\delta_F(x))$ for $x \in \mathbb{R} \setminus \{0\}$. Then the equality $(f - f \circ \tau_F)\widehat{\eta} = 0$ means that $f(x) = f(\tau_F(x))$ for a.e. $x \in G$. Now, if we take $x = \delta_{F|G}^{-1}(y)$ with $y \in F$, by using (3) we get $f(y) = f(h_{FG}(y))$ for a.e. $y \in F$. $\qquad\square$

Definition 2.5. *Let Ξ be a group of bijections of the set X and $[x]_\Xi := \{\xi(x) : \xi \in \Xi\}$, $x \in X$, be the orbit of a point x under the action of Ξ on X. A set W is called a cross-section for Ξ if W contains one and only one point from each orbit $[x]_\Xi$, $x \in X$. If Ξ is singly generated by a bijection h of X then a cross-section for Ξ will be simply called a cross-section for h.*

We have been using the term *wandering set* instead of cross-section in [2], [15]. The term wandering is used with a slightly different meaning in the theory of dynamical systems. To avoid any further confusion we decided to use a different terminology.

It is easy to see that the algebra described by (16) is nonatomic if h_{FG} admits a measurable cross-section. Let us observe then that Corollary 2.4 together with Proposition 1.1 and the above remark give, in particular, the result of Theorem 2.1 contained in [15]. A natural problem which arises in this context is to characterize the measurable isomorphisms h of F for which the algebra given as in (16) (with $h_{FG} = h$) is nonatomic. It is easy to see that in general we have the following proposition.

Proposition 2.6. *Let $(X, \mathcal{M}, \nu)$ be a measure space, $h : (X, \mathcal{M}, \nu) \to (X, \mathcal{M}, \nu)$ be a measurable bijection which preserves null sets (i.e. sets of ν measure zero) and $\mathcal{A} := \{f \in L^\infty(X, \nu) : f = f \circ h\}$. The following are equivalent:*

(i) *$\mathcal{A}$ is a nonatomic von Neumann algebra,*
(ii) *there exists a continuous map $\Omega : [0, 1] \to (\mathcal{M}, d)$ such that $\Omega(0) = \emptyset$, $\Omega(1) = X$ and $h(\Omega(t)) = \Omega(t)$, where $(\mathcal{M}, d)$ is the metric space of equivalent classes of measurable subsets of X modulo null sets with the distance $d(A, B) := \nu\left((A \backslash B) \cup (B \backslash A)\right)$, $A, B \in \mathcal{M}$,*
(iii) *there exists no measurable set of positive measure $\omega \subset X$ such that $h(\omega) \subset \omega$ and $h_{|\omega}$ is an ergodic transformation with respect to ν (i.e. for $\omega' \subset \omega$ such that $h(\omega') \subset \omega'$ it follows that either $\nu(\omega') = 0$ or $\nu(\omega \backslash \omega') = 0$).*

We should mention here that the condition on h to admit a measurable cross-section is just sufficient but not necessary to insure that $\mathcal{A}$ is nonatomic. Indeed, let us take $X = [0, 1) \times [0, 1)$, ν the product Lebesgue measure and $h(x, y) = (x, \lfloor y + \alpha \rfloor)$ with $\alpha \notin \mathbb{Q}$. Then one can choose $\Omega(t) = \{(x, y) : 0 \le x < t, y \in [0, 1]\}$ and observe that the property (ii) in Proposition 2.6 is satisfied but there exist no measurable cross-section for h since h is invariant with respect to ν.

Nevertheless, in [15] it is conjectured that every map in $\mathcal{WI}$ admits a measurable cross-section and it is shown in [2] that this is the case for a special subclass of $\mathcal{WI}$ which includes most of the known example of wavelet induced maps. One

can ask whether the condition on $h_F \in \mathcal{WI}$ to admit a measurable cross-section is also necessary in order for $\mathcal{M}_{\psi,\eta}$ to be nonatomic. For a real-valued measurable map f we use the notation $supp(f)$ for the set $\{x | f(x) \neq 0\}$.

The following corollary can be regarded as a generalization of Corollary 2.4.

Corollary 2.7. *Let ψ ne an MSF wavelet supported on F and $\eta \in \mathcal{W}(\mathcal{U})$ a wavelet having the property that $supp(\widehat{\eta}) \subset \cup_{k=1}^{m} G_k$, where $G_k \in \mathcal{WS}$, $k = 1, ..., m$. Then $\mathcal{M}_{\psi,\eta}$ is isomorphic to*

$$\{f \in L^{\infty}(F) : f(x) = (f \circ h_{FG_k})(x),$$

(17)

$$a.e. \ x \in \delta_F(supp(\widetilde{\eta}) \cap G_k) \ for \ k = 1, ..., \ m\}.$$

PROOF. As in the proof of Corollary 2.4, the result follows easily from (3) and Theorem 2.3. □

There are many wavelets η which satisfy the hypotheses of this corollary. In fact, every band-limited wavelet whose Fourier transform is a continuous function (for instance Lemarié-Meyer wavelets) are such examples. Indeed, the $supp(\widehat{w})$ is contained in a compact set which does not contain zero ([13]). Using a result proved in [17] one can always cover any such compact set with finitely many wavelet sets.

The following proposition combined with Corollary 2.7 can be regarded as a generalization of the result obtained in [15].

Proposition 2.8. *Let F, $G_k \in \mathcal{WS}$, $(k = 1, ..., m)$ and assume that the group Ξ generated by h_{FG_k} admits a measurable cross-section. Then the algebra $\{f \in L^{\infty}(F) : f = f \circ h_{FG_k} \ for \ every \ k = 1, ..., m\}$ is nonatomic. Consequently, if A_k, $(k = 1, ..., m)$ are measurable subsets of F then the algebra*

$$\{f \in L^{\infty}(F) : \quad f = f \circ h_{FG_k} \ a.e. \ on \ A_k \ for \ every \ k = 1, ..., m\}$$

is nonatomic.

PROOF. Let us denote $F_t = F \cap (-\infty, \tan(t\pi - \pi/2))$ for every $t \in [0, 1]$. Clearly, $E_0 =, E_1 = F$ and $\mu(F_t \bigtriangledown F_s) \to 0$ as $t \to s$ for $s \in [0, 1]$. Also, we denote by $\Omega_t = \bigcup_{\xi \in \Xi} \xi(F_t)$ $(t \in [0, 1])$. For every $t \in [0, 1]$ and $k = 1, ..., m$, the set Ω_t is invariant under h_{FG_k} by construction. Hence $f_t := \chi_{\Omega_t}$ has the property $f_t = f_t \circ h_{FG_k}$ for every $k = 1, ..., m$. So $\{f_t\}_{t \in [0,1)}$ is a nest of projections in our algebra joining 0 and 1. To finish the proof, we only need to show that this nest is SOT-continuous. It suffices to show that $t \to \mu(\Omega_t)$ is continuous. In order to

do so, let us observe that Ξ is a countable group and then the proof follows using the same argument as in the proof of Lemma 2.2 in [15]. $\square$

2.1. **Examples of nonatomic and atomic pairs.** The following result shows that, using our technique, every wavelet support is contained in some region of the real line can be connected by a direct path of wavelets to the Shannon wavelet. In the next theorem this region is just one of the simple extensions from E. A natural question which appears at this point is whether or not there exists a maximal region (relative to inclusion) containing E with the property that any wavelet whose support is compactly contained in this region is directly connected with the Shannon wavelet. If such a region exists it would be interesting if one could exhibit it. On the other hand, the whole real line is not such a maximal set because of Proposition 2.10.

Theorem 2.9. *Let ψ be the Shannon wavelet and $\eta \in \mathcal{W}(\mathcal{U})$ be such that $supp(\widehat{\eta}) \subset [-2\pi, -\epsilon) \cup [\pi, 4\pi - \epsilon)$ for some $\epsilon > 0$. Then ψ and η are directly connected.*

PROOF. Let us consider the following family of wavelet sets from ([3]):

$$F_\alpha = [-2\pi + 2\alpha, -\pi + \alpha) \cup [\pi + \alpha, 2\pi + 2\alpha)$$

with $\alpha \in [-\pi, \pi)$. For $n \in \mathbb{N}$ we let $\alpha_n = \pi - \frac{\pi}{2^n}$ and write $h_n = h_{F_{\alpha_n}}$. Let us observe that

$$[-2\pi, 0) \cup [\pi, 4\pi) = E \cup \bigcup_{n \in \mathbb{N}} F_{\alpha_n}$$

and so, by our hypothesis, it follows that there exists an $L \in \mathbb{N}$ such that

$$supp(\widehat{\eta}) \subset E \cup \bigcup_{n=1}^{L} F_{\alpha_n}$$

The result follows then from Corollary 2.7 and Proposition 2.8 provided we show that the group Ξ_L generated by the maps h_k $(k = 1, ..., L)$ admits a measurable cross-section. This is certainly the case if we show that Ξ_L is a finite group. One can easily find that h_k is given by

$$h_k(x) = \begin{cases} 2^{-k}x + 2\pi, & x \in [-2\pi, -\pi) \\ 2x - 4\pi, & x \in \left[\pi, \frac{3\pi}{2}\right) \\ 2x - 2\pi, & x \in \left[\frac{3\pi}{2}, 2\pi - \frac{\pi}{2^k}\right) \\ x, & x \in \left[2\pi - \frac{\pi}{2^k}, 2\pi\right). \end{cases}$$

It is more convenient to work with the wavelet induced maps $\widetilde{h}_k = \xi \circ h_k \circ \xi^{-1}$ and $\widetilde{h}_{-k} = \xi \circ h_{-k} \circ \xi^{-1}$. We write $\widetilde{\Xi}_L$ for the corresponding group generated by $\widetilde{h}_k$, $\widetilde{h}_{-k}$, $k = 1, ..., L$. It turns out that $\widetilde{h}_k$ can be given in only three pieces:

$$\widetilde{h}_k(x) = \begin{cases} \lfloor 2^{-k}(x-1) \rfloor, & x \in \left[0, \dfrac{1}{2}\right) \\ \lfloor 2x \rfloor, & x \in \left[\dfrac{1}{2}, 1 - \dfrac{1}{2^{k+1}}\right) \\ x, & x \in \left[1 - \dfrac{1}{2^{k+1}}, 1\right). \end{cases}$$

Let us denote by A the set of points $\left\{0, \dfrac{1}{2}, ..., 1 - \dfrac{1}{2^k}, ..., 1 - \dfrac{1}{2^{L+1}}\right\}$. We observe that each $\widetilde{h}_k$ leaves the set A invariant and since $\widetilde{h}_k^{(k+1)} = id$ it follows that $\widetilde{h}_k^{-1}$ leaves A invariant too. Therefore every map in $\widetilde{\Xi}_L$ leaves A invariant. Moreover, every element in $\widetilde{\Xi}_L$ is a composition of finitely many of the maps $\widetilde{h}_k$ $(k = 1, ..., L)$ and therefore it is a piecewise linear right continuous bijection of $[0, 1)$ whose points of discontinuity can only possible be in the set A. Hence every element of $\widetilde{\Xi}_L$ is uniquely determined by what it does to the set A. Therefore there cannot be more elements in $\widetilde{\Xi}_L$ then $(L + 2)!$ (the order of the permutation group of the elements of A). $\square$

It is interesting to note that a class of wavelets known as Lemarié-Meyer wavelets has been reconstructed in [3] using the natural notion of an interpolation pair of wavelets (for instance, a pair (E, F), $F \in \mathcal{WS}$, for which $h_F^2 = id$). In [11] and [12], *smoothing* of the same Shannon wavelet also led to these Lemarié-Meyer wavelets. So, there are reasons to believe that one can obtain a result similar to Theorem 2.9 which will apply to Lemarié-Meyer wavelets.

We continue with examples of pairs of wavelets which are not directly connected. Let us consider ψ be the Shannon wavelet and $\eta = \chi_{[0,1/2)} - \chi_{[1/2,1)}$ (the Haar wavelet). A simple computation shows that $\widehat{\eta}(x) = \dfrac{i}{\sqrt{2\pi}} \dfrac{(e^{ix/2}-1)^2}{x}$, $x \in \mathbb{R}\backslash\{0\}$. This map is clearly supported on the whole real line and so, the following proposition shows that ψ and φ cannot be connected by a direct path.

Proposition 2.10. *Let ψ be the Shannon wavelet and $\eta \in \mathcal{W}(\mathcal{U})$ be such that $supp(\widehat{\eta}) \supseteq [-4\pi, -2\pi] \cup [2\pi, 4\pi]$. Then $\mathcal{M}_{\psi,\eta} = \mathbb{C}I$ and so ψ and η are not directly connected.*

PROOF. Let $f \in L^\infty(E)$ such that $M_{\widetilde{f}} \in \widehat{\mathcal{M}_{\psi,\eta}}$ where $\widetilde{f}$ is defined by (9) in the case $F := E$. Using Theorem 2.3, we obtain that $f(\delta_E(x)) = f(\tau_E(x))$ a.e.

$x \in 2E$ since by hypothesis $2E \subset \operatorname{supp}(\widehat{\eta})$. Hence, we have that

(18) $f(x) = f(\delta_E(2x)) = f(\tau_E(2x))$ a.e. $x \in E$.

We write $\widehat{f} = f \circ \xi^{-1}$ where ξ was defined in (4). It is clear that $\widehat{f} \in L^\infty([0,1))$ and (18) becomes

$$\widehat{f}(x) = \widehat{f}(\lfloor 2x \rfloor) \ a.e. \ x \in [0,1),$$

where by $\lfloor x \rfloor$ we denoted the fractional part of the real number x. In [25] it was proved that the transformation $T(x) = \lfloor 2x \rfloor$ is an ergodic transformation on $[0,1)$ with respect to an invariant measure ν which is equivalent to Lebesgue measure. By ergodicity theorem (see for instance Theorem 4.4 in [24]), for every $g \in L^1([0,1),\nu)(= L^1([0,1),\mu))$ we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} g(T^k(x)) = \int_{[0,1)} g d\nu \ a.e. [\nu]([\mu]) \ x \in [0,1).$$

Since $\widetilde{f} = \widetilde{f} \circ T$ in $L^\infty([0,1))$ it follows that f is essentially just a constant function. Hence, we get that $\widehat{\mathcal{M}_{\psi,\eta}} = \mathbb{C}I$. $\square$

Remark. It was proved in [20] that every piecewise C^1 function f on $[0,1)$ such that $\inf |f'| > 1$ admits an invariant measure which is absolutely continuous with respect to Lebesgue measure. In [21](see also [18]), the authors improved upon this result and showed that there exist finitely many sets L_j $(j = 1, ..., n)$ (n is the number of discontinuities of f and/or f') such that f is ergodic on L_j with respect to the Lebesgue measure. This result can be used to prove the following more general result than Proposition 2.10.

Proposition 2.11. *Let ψ be the Shannon wavelet and $\eta \in \mathcal{W}(\mathcal{U})$ be such that $\operatorname{supp}(\widehat{\eta})$ contains a finite union of intervals $\bigcup_k J_k \subset (-\infty, -2\pi) \cup [2\pi, \infty)$ with the property that $\delta_E \left(\bigcup_k J_k\right) = E$. Then $\mathcal{M}_{\psi,\eta} \subset \mathbb{C}^m$ for some $m \in \mathbb{N}$ and so ψ and η are not directly connected.*

3. The n-dimensional case

Let $\mathbb{R}^n$ be, as usual, n-dimensional Euclidian space and take $H = L^2(\mathbb{R}^n)$ the complex Hilbert space of (equivalence classes of) square integrable complex-valued functions on $\mathbb{R}^n$ relative to Lebesgue-Borel measure μ_n on $\mathbb{R}^n$. The Fourier transform of $f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ is defined by

$$\widehat{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i<x,t>} f(t) d\mu_n(t).$$

The unitary systems pertinent to orthonormal wavelet theory in higher dimensions arise in the following way. For $k \in \mathbb{N}$, let $M_k(\mathbb{R})$ denote the algebra of $k \times k$ matrices with entries from $\mathbb{R}$. If $A \in M_n(\mathbb{R})$ is invertible and $v \in \mathbb{R}^n$ then the operators D_A, T_v in $B(L^2(\mathbb{R}^n))$ defined by

$$(D_A f)(x) = |\det A|^{\frac{1}{2}} f(Ax) \ (a.e.) \qquad x \in \mathbb{R}^n, \qquad f \in L^2(\mathbb{R}^n),$$

$$(T_v f)(x) = f(x - v) \ (a.e.) \qquad x \in \mathbb{R}^n, \qquad f \in L^2(\mathbb{R}^n),$$

are clearly unitary operators. Let $e_1, e_2, ..., e_n$ be the standard basis for $\mathbb{R}^n$ and let $L := \{x \in \mathbb{R}^n : x = \sum_{k=1}^n x_k e_k, \ x_k \in \mathbb{Z}\}$ be the group of vectors generated by integer linear combinations of the basis $\{e_k\}$. Then we set

$$\mathcal{U}(= \mathcal{U}_A) := \{D_A^k T_v : \ k \in \mathbb{Z}, \ v \in L\}.$$

If $f \in L^2(\mathbb{R}^n)$ is a complete wandering vector for the above unitary system, then f is called a (single-function) *dilation-A orthonormal wavelet*, and the collection of all such wavelets relative to this unitary system will be denoted by $\mathcal{W}_A$. One can consider that $\{e_k\}$ is just an arbitrary basis of $\mathbb{R}^n$ but it was shown in [16] that we can always change the matrix A accordingly to obtain an equivalent wavelet theory using the standard basis. In fact, two such unitary systems generated by matrices A and B are weakly equivalent (in the sense of [16]) if and only if $A = CBC^{-1}$ where C is a matrix with integer coefficients and determinant ± 1.

It is known [4] that wavelets exist when A is an *expansive matrix* (i.e. all the eigenvalues of the matrix A have modulus greater than 1). Various examples of dilation-A wavelets corresponding to different matrices A of interest are explicitly constructed in [5] and [9]. All of this examples are (MSF)-wavelets (i.e. the absolute value of their Fourier transform is the characteristic function of a measurable set).

Most of the facts from one-dimensional case can be generalized to this setting. First we need the following lemma.

Lemma 3.1. *Let $A \in M_n(\mathbb{R})$ be an expansive matrix and L be the group of vectors in $\mathbb{R}^n$ defined above. Then the set $\{A^k v : v \in L, \ k \in \mathbb{Z}\}$ is dense in $\mathbb{R}^n$.*

PROOF. Clearly, $\lim_{m \to \infty} A^{-m} e_k = 0$ for every $k = 1, ..., n$. Hence, given an $\epsilon > 0$ there exist $m \in \mathbb{N}$ such that $\|A^{-m} e_k\| < \epsilon$ for all $k = 1, ..., n$. For $u \in \mathbb{R}^n$ let $A^m u = \sum_{k=1}^n u_k e_k$ be the writing of $A^m u$ in the basis $\{e_k\}$. Then, if we denote by $[x]$ the greatest integer smaller than the real number x, we have

$$\|u - A^{-m}(\sum_{k=1}^n [u_k] e_k)\| = \|A^{-m}(\sum_{k=1}^n (u_k - [u_k]) e_k)\| \leq \sum_{k=1}^n \lfloor u_k \rfloor \|A^{-m} e_k\| \leq n\epsilon,$$

which proves our lemma.   □

An interesting problem which arises naturally is to characterize all matrices A with the property that $\{A^k v : v \in L, \ k \in \mathbb{Z}\}$ is dense in $\mathbb{R}^n$. Next, we state and sketch the proof of the multi-dimensional equivalent of (8).

Proposition 3.2. *Let $A \in M_n(\mathbb{R})$ be an expansive matrix and $\mathcal{U}_A$ defined as above. Then*

$$\widehat{\mathcal{U}'_A}\left(= \{\widehat{T} \in B(L^2(\mathbb{R}^n)) : \widehat{T}S = S\widehat{T}, \ S \in \widehat{\mathcal{U}_A}\}\right) =$$

(19)

$$\{M_f \in B(L^2(\mathbb{R}^n)) : f \in L^\infty(\mathbb{R}^n), \ \ f(x) = f(A^t x) \ a.e. \ x \in \mathbb{R}^n\}$$

where A^t is the transpose of the matrix A.

PROOF. The proof follows exactly the same steps as in the one-dimensional case and we include it here just for completeness. It easy to see that $D_{A^{-m}} T_v D_{A^m} = T_{A^m v}$ for every $m \in \mathbb{Z}$ and $v \in L$. Then if $T \in \mathcal{U}'_A$ then T commutes with all the translation operators of the form $T_{A^m v}$ ($m \in \mathbb{Z}$, $v \in L$) and then by Lemma 3.1, T commutes with all the translations T_u, $u \in \mathbb{R}^n$. Let $K := \underbrace{[0,1) \times ... \times [0,1)}_{n \text{ times}}$ be the generalized cube in $\mathbb{R}^n$. It is easy to see that the vector χ_K is cyclic for the abelian von Neumann algebra generated by all translation operators which is denoted by $\mathcal{T}$. Hence, $\mathcal{T}$ is a m.a.s.a (maximal abelian selfadjoint algebra). It follows that $\mathcal{U}'_A$ is contained in $\mathcal{T}' = \mathcal{T}$. A simple computation shows that $\widehat{D_{A^{-1}}} = D_{A^t}$ and $\widehat{T_u} = M_{e^{-i<\cdot,u>}}$ for every $u \in \mathbb{R}^n$. Hence, $\widehat{\mathcal{T}}$ is a m.a.s.a and it is generated by the multiplication operators $M_{e^{-i<\cdot,u>}}$. Since the algebra $\mathcal{M}$ of all multiplication operators with functions in $L^\infty(\mathbb{R}^n)$ contains $\widehat{\mathcal{T}}$ and it is a m.a.s.a itself, it follows that $\widehat{\mathcal{T}} = \mathcal{M}$. Now, for $f \in L^\infty(\mathbb{R}^n)$, it is clear that M_f is in $\widehat{\mathcal{U}'_A}$ if and only if it commutes with $\widehat{D_A}$, or equivalently, it commutes with D_{A^t}. Since $M_f D_{A^t} = D_{A^t} M_f$ is equivalent to $f(x) = f(A^t x)$ for a.e. $x \in \mathbb{R}^n$ the proposition is proved.   □

As in the one-dimensional case and following the terminology in [4] and [5], we say that a subset F of $\mathbb{R}^n$ of positive measure is a *dilation-A wavelet set* if the inverse Fourier transform of $(\mu_n(F))^{\frac{-1}{2}} \chi_F$ is a dilation-A orthonormal wavelet.

It is shown in [4] that a set F is an wavelet set if and only if

$$\{(A^t)^k(F)\}_{k \in \mathbb{Z}} \quad \text{and} \quad \{T_u(W)\}_{u \in L}$$

are both partitions for $\mathbb{R}^n$ (modulo sets of measure zero).

In a proof similar to that given in [15] for $n = 1$, it can be shown and we will assumed that F is a regularized wavelet set i.e. $\{(A^t)^k(F)\}_{k \in \mathbb{Z}}$ is a veritable partition of $\mathbb{R}^n \backslash \{0\}$ and $\{T_u(F)\}_{u \in L}$ is a genuine partition of $\mathbb{R}^n$. We will simply denote by $\mathcal{WS}(n, A)$ the collection of all dilation-A regularized wavelet sets. In what follows we will fix $F \in \mathcal{WS}(n, A)$. It is then natural to define the maps $\delta_F : \mathbb{R}^n \backslash \{0\} \to F$, $\tau_F : \mathbb{R}^n \to F$ by setting $\delta_F(x) = (A^t)^{k(x)}x$ $(x \in \mathbb{R}^n \backslash \{0\})$ and $\tau_F(x) = T_{u(x)}x$ $(x \in \mathbb{R}^n)$, where $k(x) \in \mathbb{Z}$ and $u(x) \in L$ are uniquely determined by the conditions $(A^t)^{k(x)}x \in F$ and $T_{u(x)}x \in F$.

If G is another regularized wavelet set then $\delta_{F|G} : G \to F$ and $\tau_{F|G} : G \to F$ are measurable bijections and we can define the measurable bijection $h_{FG} : F \to F$ analogously to the case $n = 1$ by

$$(20) \qquad h_{FG} := \tau_{F|G} \circ \delta_{F|G}^{-1}$$

Clearly by Proposition 3.2, $\widehat{\mathcal{U}}$ is isomorphic to the algebra $L^\infty(F)$ via the isomorphism $f \to M_{\widetilde{f}}$ where

$$(21) \qquad \widetilde{f}(x) = f(\delta_F(x)).$$

Let us fix $\psi, \eta \in \mathcal{W}_A$ and $S \in \{\mathcal{U}\}'$. Then $S \in \{\mathcal{U}, V_\psi^\eta\}'$ if and only if $SV_\psi^\eta = V_\psi^\eta S$. If we take into account that $\{D_A^k T_u \psi\}_{k \in \mathbb{Z}, u \in L}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$ we get that $S \in \{\mathcal{U}, V_\psi^\eta\}'$ if and only if

$$(22) \qquad S(D_A^k T_u \eta) = V_\psi^\eta D_A^k T_u S \psi, \quad k \in \mathbb{Z}, \ u \in L.$$

We write $S\psi$ in the basis $\{D_A^k T_u \psi\}_{k \in \mathbb{Z}, u \in L}$ as $S\psi = \sum_{k \in \mathbb{Z}, u \in L} a_{k,u} D_A^k T_u \psi$ and then the equation (22) becomes

$$(23) \qquad S(D_A^k T_u \eta) = \sum_{l \in \mathbb{Z}, v \in L} a_{l,v} V_\psi^\eta D_A^k T_u D_A^l T_v \psi, \quad k \in \mathbb{Z}, \ u \in L.$$

One can check that we do have a similar relations to those in (7)

$$T_v T_w = T_{v+w}, \quad D_B D_C = D_{CB}, \quad T_v D_B = D_B T_{Bv}, \quad B, C \in M_n(\mathbb{R}), \ v, w \in \mathbb{R}^n.$$

Using these properties, (23) becomes

$$(24) \qquad S(D_A^k T_u \eta) = \sum_{l \in \mathbb{Z}, v \in L} a_{l,v} V_\psi^\eta D_A^{k+l} T_{A^l u + v} \psi, \quad k \in \mathbb{Z}, \ u \in L.$$

Now, it becomes clear that if we want to obtain similar results as in case $n = 1$ we need to make new assumptions on the matrix A. As in some other papers on n-dimensional wavelets we will assume that $A(L) \subset L$ (see for instance [9]). (We observe that the equation (24) simplifies even more if we assume in addition that $A^{-1}(L) \subset L$, which it is satisfied let's say if A has integer coefficients and

determinant 1 or -1. Unfortunately, for dilation matrices of this type, since they are not expansive, there is no guaranty that the set $\mathcal{W}_A$ is not empty.) Hence, we can rewrite (24) as

(25)
$$S(D_A^k T_u \eta) = \sum_{l \in \mathbb{Z}, l \geq 0, v \in L} a_{l,v} D_A^{k+l} T_{A^l u + v} \eta +$$

$$\sum_{l \in \mathbb{Z}, l < 0, v \in L} a_{l,v} V_\psi^\eta D_A^{k+l} T_{A^l u + v} \psi, \quad k \in \mathbb{Z}, \ u \in L.$$

These equations simplify considerably if we assume that ψ is a (MSF)-dilation-A wavelet defined by $|\widehat{\psi}| = (\mu_n(F))^{\frac{-1}{2}} \chi_F$. Since $S \in \mathcal{U}'$, by Proposition 3.2, there exists $f \in L^\infty(F)$ such that $\widehat{S} = M_{\widetilde{f}}$. It is easy to see that the family of functions $\{e_u\}_{u \in L}$ $(e_u(x) = (\mu_n(F))^{\frac{-1}{2}} e^{i<x,u>}, x \in F)$ forms an orthonormal basis for $L^2(F)$. Then if $f(x) = \sum_{u \in L} c_u e^{i<x,u>}$ in $L^2(F)$, we have as before

(26)
$$S\psi = \sum_{u \in L} c_u T_u \psi.$$

Therefore, using (25) for this case, it follows that $S \in \{\mathcal{U}, V_\psi^\eta\}'$ if and only if

(27)
$$S(D_A^k T_u \eta) = \sum_{v \in L} c_v D_A^k T_{u+v} \eta, \quad k \in \mathbb{Z}, \ u \in L.$$

The following theorem generalizes Theorem 2.3.

Theorem 3.3. *Let A be a $n \times n$ matrix with real coefficients which is expansive and satisfies the property $A(\mathbb{Z}^n) \subset \mathbb{Z}^n$. Consider ψ a (MSF)-dilation-A wavelet supported on $F \in \mathcal{WS}(n, A)$ and let $\eta \in \mathcal{W}_A$. Then*

(i) *The algebra $\mathcal{M}_{\psi, \eta}$ is isomorphic to the following subalgebra of $L^\infty(F)$*

$$\mathcal{A} := \{f \in L^\infty(F) : \quad (f \circ \delta_F - f \circ \tau_F)\widehat{\eta} = 0)\};$$

(ii) *If $|\widehat{\eta}| = (\mu_n(F))^{\frac{-1}{2}} \chi_G$ where $G \in \mathcal{WS}(n, A)$ we have $\mathcal{A} = \{f \in L^\infty(F) : f = f \circ h_{FG}\}$.*

The proof of this theorem uses (27) and the same idea as in the proof of Theorem 2.3.

REFERENCES

[1] E. Azoff, *On finite rank operators and preannihilators*, Memoirs Amer. Math. Soc. **64**(1986).

[2] E. Azoff and E. Ionascu, *Wandering sets for a class of Borel isomorphisms of* $[0, 1)$, J. Fourier Anal. Appl., **6**(2000), 623–638.

[3] X. Dai and D. Larson, *Wandering vectors for unitary systems and orthogonal wavelets*, Memoirs Amer. Math. Soc. **134**(1998).

[4] X. Dai, D. Larson and D. Speegle, *Wavelet sets in* $\mathbb{R}^n$, J. Fourier Anal. Appl. **3**(1997), 451–456.

[5] X. Dai, D. Larson and M. Speegle, *Wavelet sets in* $\mathbb{R}^n$*. II.*, *Wavelets, multiwavelets, and their applications* (*San Diego, CA, 1997*), 15–40, Contemp. Math. **216**, Amer. Math. Soc., Providence, RI, 1998.

[6] X. Dai and R. Liang, *Path Connectivity and wavelets*, unpublished manuscript.

[7] X. Dai and S. Lu, *Wavelets in subspaces*, Michigan Math. J. **43**(1996), 81–98.

[8] Q. Gu, *Wavelets and wavelet sets*, Thesis, 1998.

[9] Q. Gu and D. Han, *On multiresolution analysis (MRA) wavelets in* $\mathbb{R}^n$, J. Fourier Anal. Appl. **6**(2000), 437–447.

[10] D. Han and S. Lu, *Smoothing approximations and pathwise connectivity of MRA wavelets*, unpublished manuscript.

[11] E. Hernandez, X. Wang and G. Weiss, *Smoothing minimally supported frequency wavelets. I.*, J. Fourier Anal. Appl. **2**(1996), 329–340.

[12] E. Hernandez, X. Wang and G. Weiss, *Smoothing minimally supported frequency wavelets. II.*, J. Fourier Anal. Appl. **3**(1997), 23–41.

[13] E. Hernandez and G. Weiss, *A first course on wavelets*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1996.

[14] E. Ionascu, *A new construction of wavelet sets*, to appear.

[15] E. Ionascu, D. Larson and C. Pearcy, *On wavelet sets*, J. Fourier Anal. and Appl. **4**(1998), 711–721.

[16] E. Ionascu, D. Larson, and C. Pearcy, *On the unitary systems affiliated with orthonormal wavelet theory in n-dimensions*, J. Funct. Anal. **157**(1998), 413–431.

[17] E. Ionascu and C. Pearcy, *On subwavelet sets*, Proc. Amer. Math. Soc. **126**(1998), 3549–3552.

[18] A. Kosjakin and E. Sandler, *Ergodic properties of a certain class of piecewise smooth transformations of the segment*, Izv. Vysš Učebn. Zaved. Matematika, **118** (1972), 32-40.

[19] D. Larson, *Von Neumann algebras and wavelets*, Operator algebras and applications (Samos, 1996), 267–312, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **495**, 267–312, Kluwer Acad. Publ., Dordrecht, 1997.

[20] A. Lasota and J. Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. **186** (1973), 481–488.

[21] T. Li and J. Yorke, *Ergodic transformations from an interval into itself*, Trans. Amer. Math. Soc. **253**(1978), 183–192.

[22] J. Oxtoby, *Measure and category*, Springer-Verlag, New York, 1970.

[23] W. Parry, *On the β-expansions of real numbers* (English. Russian summary) Acta Math. Acad. Sci. Hungar. **11** (1960), 401–416.

[24] K. Petersen, *Ergodic theory*, Cambridge University Press, Cambridge, 1983.

[25] A. Rényi, *Representations for real numbers and their ergodic properties*, (English) Acta Math. Acad. Sci. Hungar. **8** (1957), 477–493.

[26] D. Speegle, *The s-elementary wavelets are path-connected*, Proc. Amer. Math. Soc. **127** (1999), 223–233.

[27] WUTAM Consortium, *Basic properties of wavelets*, J. Fourier Anal. Appl. **4**(1998), 575-594.

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